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Age Replacement with Discounting for a Continuous Maintenance Cost Model

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The optimal age replacement policy according to the criterion of Fox [5] is obtained for the model, introduced by Scheaffer [7], containing a continuously increasing maintenance cost factor. The new results beyond the generalisation to a new model are mainly in sensitivity analysis. Examples are given where the optimal replacement age increases with the interest rate, as well as examples of the opposite.

KEY WORDS

Age Replacement
Discounting
Maintenance Cost

SUMMARY OF NOTATION

The total discounted expense for keeping successive units of some machine operating indefinitely is obtained when replacement takes place at failure or at age T, whichever occurs first.

C_1 = cost for replacing unit which has failed
C_2 = cost for replacing unit which has been replaced at age T
g = maintenance cost intensity
p = interest rate ($100p$ % per year)
$\delta = \ln(1 + p)$, discount rate
F = unit life distribution function
$r(x) = F'(x)/(1 - F(x))$, failure rate function
$\varphi(x) = (C_1 - C_2)r(x) + g(x)$
$a(x) = e^{-\delta x}(1 - F(x))$
$A(x) = \int_0^x a(u)\, du$
$H(T) = \dfrac{\int_0^T \varphi(x)a(x)\, dx + C_2}{A(T)}$, objective function to be minimized
$\psi(T) = \int_0^T [\varphi(T) - \varphi(x)]a(x)\, dx - C_2$; except for a positive factor this is $H'(T)$
$\hat{T}$ = solution to min $H(T)$; $H(T) \geq H(\hat{T})$ for $T > 0$.

1. INTRODUCTION

We are concerned with the problem of age replacement policies where replacement of a unit occurs at failure or at age T, whichever occurs first. In most previous studies the optimum problem has been defined as finding T which minimizes the average expected cost per unit time over an infinite time span. This implies the assumption that an expense C has the same value regardless of when it is incurred. In business practice it is more common to discount an expense C suffered at time t with a factor $(1 + p)^{-t}$ to obtain its value at time 0. Here $100p$ % is the internal interest rate or yield on capital, taking the year as the unit of time.

In this paper we propose to obtain the optimal T which minimizes the expectation of the sum of discounted expenses when the planning horizon is infinite, thus following the method of Fox [5], [6] and Denardo & Fox [3]. We make the following assumptions.

The unit lives $X_1, X_2, \cdots$ are IID random variables distributed as X with an absolutely continuous distribution function F such that $F(0) = 0$ and $F(x) < 1$. We define

$$r(x) = F'(x)/(1 - F(x)),$$

the failure rate function. In most applications it is reasonable to assume that this is increasing (IFR). The interest rate p is constant, and we put

$$\delta = \ln(1 + p) > 0.$$

A cost C_1 is incurred for each failed unit which is replaced, and a cost $C_2 < C_1$ is incurred for each non-failed unit which is exchanged. Further a continuously increasing cost is incurred for maintenance of a unit, so that

$$\int_0^t g(u)\, du$$

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has been paid for a unit of age t. This generalises Fox's model, which is obtained by putting $g(x) = 0$.

We may more generally think of the maintenance cost suffered up to age t as an increasing stochastic process $G(t)$ independent of unit age X and with intensity function g, for example a Poisson process with

$$E(G(t)) = \int_0^t g(u)\, du.$$

This will lead to the same expression for the mean discounted cost.

The value of money should be taken as constant. The assumption of constant p and δ is not a serious restriction, since economic studies have indicated that real interest rates with the effect of inflation eliminated are rather constant.

It is well known from the theory of investment calculus that the present optimality criterion is the only one consistent with long run profit maximizing in societies with positive interest rates. Basically it is a question of whether a resource available only at some future time is worth as much as the same resource available now. There are reasons to believe that the need to reduce the value of future resources is politically neutral and recognized in socialist as well as capitalist countries.

As for the model, it is of course a simplification of real life situations. Specifically, it does not allow for an age-dependent secondhand value of the equipment. This would be allowed for by letting the replacement costs depend on equipment age t at replacement, putting $C_i = C_i(t)$, $i = 1, 2$. It is easy to obtain the expected sum $E(Y)$ of discounted expenses for this more general model; we only have to replace C_1 with $C_1(x)$ and C_2 with $C_2(T)$ in the expressions $\varphi(x)$, $E(Y)$ and $H(T)$ (see Eq. (12)). The general results and insights of the subsequent analysis would, however, be obscured by complications like this. For this reason, and since there are reasons to believe that our assumptions are adequate approximations to reality in many situations, we have chosen to employ the same (basically) model as previous authors.

The introduction of a maintenance cost should make the model more realistic in practical work. See Scheaffer [7], Cléroux & Hanscom [2] and Tilquin & Cléroux [8]. In [2] and [8] the authors considered a discontinuous maintenance cost function instead of the integral above; such a model may however be arbitrarily well approximated by ours, since one may let g be very large on very small intervals. The advantage of a cost intensity is of course the greater mathematical tractability.

We will derive the expected discounted expense for keeping successive units operating indefinitely as a function of the replacement age T. The problem is to minimize this function. In the limit as $\delta \to 0$ this is the same as the traditional problem (Proposition 1). Sufficient conditions are given for the existence of a finite global minimum $\hat{T}$ (Proposition 2, Eq. (6)). A useful delimitation of possible values for $\hat{T}$ is also given (Proposition 3). In this we generalise and strengthen the results of Fox [5] and Denardo & Fox [3]. In sensitivity analysis no previous work has been done. Here we study mainly the dependence of $\hat{T}$ on δ. A natural condition implying that $\hat{T}$ increases in δ is obtained (Proposition 4), and further we discuss conditions under which $\hat{T}$ may decrease in δ. Examples illustrating both these possibilities are finally given.

The dependence of $\hat{T}$ on δ seems to be particularly interesting from the viewpoint of econometrics and forecasting. It is usually assumed that a decrease in interest rates entails more capital expenditure. While this is so under the conditions of Proposition 4, our examples show that the opposite might be the case for certain types of equipment.

2. Value of Discounted Expense

Let us start with a new unit at time $0 = t_0$. Units are replaced at times $t_1, t_2, \cdots$. With

$$Z_i = t_i - t_{i-1},$$

the $Z_i = \min(X_i, T)$ are IID and distributed as $Z = \min(X, T)$. We have

$$t_i = \sum_{k=1}^{i} Z_k .$$

With

$$C(u) = \begin{cases} C_1, & \text{if } u < T \\ C_2, & \text{if } u \geq T, \end{cases}$$

the cost for maintenance and replacement of the i:th unit, located to the time of replacement t_i, is

$$C(Z_i) + \int_0^{Z_i} g(u)(1 + p)^{Z_i - u}\, du.$$

The total cost Y for maintenance and replacement, located to time 0, is then the sum of all these terms, each discounted with the factor $(1 + p)^{-t_i}$;

$$(1) \qquad Y = \sum_{i=1}^{\infty} \left[C(Z_i) + \int_0^{Z_i} g(u) e^{\delta(Z_i - u)}\, du \right] \cdot \exp\left(-\delta \sum_{k=1}^{i} Z_k\right).$$

Here Y is a random variable. All its terms are positive, so by monotone convergence,

$$E(Y) = \sum_{i=1}^{\infty} E\left[\left(\prod_{k=1}^{i-1} e^{-\delta Z_k}\right)\left\{C(Z_i)e^{-\delta Z_i} + \int_0^{Z_i} g(u)e^{-\delta u}\,du\right\}\right]$$

$$= \sum_{i=1}^{\infty} E(e^{-\delta Z})^{i-1}\left\{E(C(Z)e^{-\delta Z}) + E\left(\int_0^{Z} g(u)e^{-\delta u}\,du\right)\right\}$$

$$= \left[E(C(Z)e^{-\delta Z}) + E\left(\int_0^{Z} g(u)e^{-\delta u}\,du\right)\right]/(1 - E(e^{-\delta Z})).$$

Thus $E(Y) < \infty$ and Y is finite with probability one. We see that $E(Y) \to \infty$ as $\delta \to 0$ and $E(Y) \to 0$ as $\delta \to \infty$. We have

$$E(C(Z)e^{-\delta Z}) = \int_0^T C_1 e^{-\delta x} F'(x)\,dx + C_2 e^{-\delta T}(1 - F(T));$$

$$E\left(\int_0^Z g(u)e^{-\delta u}\,du\right) = \int_0^T F'(x)\int_0^x g(u)e^{-\delta u}\,du\,dx + (1 - F(T))\int_0^T g(u)e^{-\delta u}\,du$$

$$= \int_0^T (1 - F(x))g(x)e^{-\delta x}\,dx$$

and

$$E(e^{-\delta Z}) = \int_0^T e^{-\delta x}F'(x)\,dx + (1 - F(T))e^{-\delta T}.$$

Define

$$\varphi(x) = (C_1 - C_2)r(x) + g(x).$$

After some manipulations we obtain

$$E(Y) = \delta^{-1}H(T) - C_2, \tag{2}$$

where

$$H(T) = \frac{\int_0^T \varphi(x)e^{-\delta x}(1 - F(x))\,dx + C_2}{\int_0^T e^{-\delta x}(1 - F(x))\,dx}.$$

We want to minimize $H(T)$ as function of T, and this has a meaning also for $\delta = 0$, even though $Y = \infty$ for zero interest. In fact we have (cf. Fox [5], Theorem 3)

Proposition 1.

$\lim_{\delta\to 0} \delta E$ (total discounted cost with discount rate δ) $= \lim_{t\to\infty} t^{-1}E$ (non-discounted cost incurred up to time t)

Proof. The method of Fox applies also to the generalised model. Define the measure M on $(0, \infty)$ by $M\{I\} = E$ (non-discounted cost incurred in I); then of course

$$E(Y) = \int_0^\infty e^{-\delta x}M\{dx\}.$$

From (2) it is seen that the left side limit exists and is positive, and so by a Tauberian theorem (Feller [4], p. 445) the right side limit exists and is equal to the left side. ■

For the particular forms for g treated by Scheaffer we can verify directly that $H(T)$ for $\delta = 0$ is Scheaffer's objective function.

In the sequel we shall admit also the δ-value 0, the minimum problem then interpreted as the traditional time-average problem.

Note that for purposes of analysis the function H is in a more convenient form than the objective functions of Fox and Scheaffer, even though the present model is more general. Fox's objective function R is, for $g(x) = 0$, related to ours by $R(T) = \delta^{-1}H(T) - C_2$.

3. Minimizing $H(T)$

H is continuous and, at points where φ is continuous, differentiable. At points where φ is discontinuous H has corners, some of which may be local minima. Other possible local minima are points where $H'(T) = 0$. To find these we differentiate H. Put

$$a(x) = e^{-\delta x}(1 - F(x)) = \exp\left\{-\left(\delta x + \int_0^x r(u)\,du\right)\right\};$$

$$A(x) = \int_0^x a(u)\,du;$$

$$\psi(T) = \int_0^T [\varphi(T) - \varphi(x)]a(x)\,dx - C_2.$$

We find

$$H'(T) = a(T)A(T)^{-2}\psi(T) = a(T)A(T)^{-1}[\varphi(T) - H(T)]; \tag{3}$$

$$H(T) = \varphi(T) - A(T)^{-1}\psi(T) = \varphi(T) - a(T)^{-1}A(T)H'(T). \tag{4}$$

Since we assume $F(x) < 1$ (otherwise we confine the search for a minimum to $(0, \inf\{x: F(x) = 1\})$) we have

$$H'(T) = 0 \Leftrightarrow \psi(T) = 0 \Leftrightarrow H(T) = \varphi(T). \tag{5}$$

This result is partially contained in Fox [5], Theorem 1.

The problem is now to establish conditions securing the existence of a finite global minimum $\hat{T}$.

Proposition 2. Assume there is a $T_0 < \infty$ such that $\varphi(T) > \min \{H(x): 0 < x \leq T_0\}$ for $T > T_0$. Then $\hat{T} \leq T_0$.

Proof. Put $c = \min \{H(x): 0 < x \leq T_0\}$. Then $H(T_0) \geq c$ implies

$$\int_0^{T_0} (\varphi(x) - c)a(x)\, dx + C_2 \geq 0.$$

For $T > T_0$ we have

$$H(T) - c = A(T)^{-1}\left[\int_0^{T_0} (\varphi(x) - c)a(x)\, dx + C_2 + \int_{T_0}^{T} (\varphi(x) - c)a(x)\, dx\right] > 0.$$

Hence $\hat{T} \leq T_0$, and there is no other global minimum in (T_0, ∞). Moreover $\delta > 0$ or $E(X) < \infty$ implies $A(\infty) < \infty$ so that $\lim_{T\to\infty} H(T) > c$ and $\hat{T} \neq \infty$. ■

A quick corollary is

$$(6) \qquad \varphi(T) \to \infty \quad \text{implies} \quad \hat{T} < \infty.$$

This strengthens Theorem 1 of Fox [5], who assumes r continuous and strictly increasing to ∞.

In [3] Denardo & Fox show that, for $g(x) = 0$, $\hat{T}$ cannot belong to an interval of decreasing failure rate. In the present setup this follows easily from the form of ψ. Since ψ increases (decreases) as φ increases (decreases), and since $\psi'(T)$ exists if and only if $\varphi'(T)$ exists, in which case $\psi'(T) = A(T)\varphi'(T)$ we have

Proposition 3. $\hat{T}$ belongs to no open interval where φ is decreasing and not constant. Further $\varphi'(\hat{T}) \geq 0$ if existing.

Some cases that might typically be encountered are

Case 1. φ increasing. Then ψ increases and may or may not cross the T-axis, the latter case corresponding to $\hat{T} = \infty$. If φ increases strictly there is only one minimum, otherwise we might obtain $\hat{T}$ as any point in a closed interval. Fox [5] states mistakenly that there may be more than one local minimum when $g(x) = 0$ and r increases strictly to ∞, and gives an unnecessary (it would seem) algorithm for finding the global minimum ([5], Theorem 2).

Case 2. φ U-shaped, caused by adjustments at the beginning of the operation of a unit. Thus ψ starts at $-C_2$ and is U-shaped, so this is the same problem as Case 1, but with a translation of the origin to the minimum point for φ and ψ.

Note that if φ has jumps H has corners at these points, which might be local minima (namely if ψ jumps from below the T-axis to above it) even though $H'(T)$ does not exist.

4. Sensitivity Analysis

The parameters C_1, C_2, δ and the functions F and g determine $\hat{T}$. If we calculate the partial derivatives of the relation $\psi(\hat{T}) = 0$ with respect to the parameters (subject to sufficient regularity conditions) we get an idea of the sensitivity of the model to changes in parameters. If we write the maintenance cost intensity as

$$g(x) = C_3 g_0(x),$$

then we may regard C_1, C_2, C_3 and δ as cost parameters determined by economic relations in the outside world, while g_0 and F are inherent to the technology of the equipment.

Assume now r and g continuous and differentiable at $\hat{T}$. Then $\varphi'(\hat{T}) \geq 0$ (Proposition 3), but we assume moreover that $\varphi'(\hat{T}) > 0$. Also assume that the global minimum is unique. Then

$$(7)\quad \begin{cases} \dfrac{\partial \hat{T}}{\partial C_1} = -\left(\displaystyle\int_0^{\hat{T}} [r(\hat{T}) - r(x)]a(x)\, dx\right)(\varphi'(\hat{T})A(\hat{T}))^{-1}; \\ \dfrac{\partial \hat{T}}{\partial C_2} = \left(1 + \displaystyle\int_0^{\hat{T}} [r(\hat{T}) - r(x)]a(x)\, dx\right)(\varphi'(\hat{T})A(\hat{T}))^{-1}; \\ \dfrac{\partial \hat{T}}{\partial C_3} = -\left(\displaystyle\int_0^{\hat{T}} [g_0(\hat{T}) - g_0(x)]a(x)\, dx\right)(\varphi'(\hat{T})A(\hat{T}))^{-1}; \\ \dfrac{\partial \hat{T}}{\partial \delta} = \left(\displaystyle\int_0^{\hat{T}} x[\varphi(\hat{T}) - \varphi(x)]a(x)\, dx\right)(\varphi'(\hat{T})A(\hat{T}))^{-1}. \end{cases}$$

Representing the change in $\hat{T}$ as

$$\Delta\hat{T} = \frac{\partial \hat{T}}{\partial C_1}\Delta C_1 + \frac{\partial \hat{T}}{\partial C_2}\Delta C_2 + \frac{\partial \hat{T}}{\partial C_3}\Delta C_3 + \frac{\partial \hat{T}}{\partial \delta}\Delta\delta + o((\Delta C_1^2 + \Delta C_2^2 + \Delta C_3^2 + \Delta\delta^2)^{1/2})$$

we get a rough idea of the performance of our model due to small changes in the cost parameters.

Now $\hat{T}$ may not even be a continuous function of (C_1, C_2, C_3, δ) (see examples). But if the dependence of $\hat{T}$ on the parameters is determined by (7), then it is monotone under general conditions;

$$1 + \int_0^{\hat{T}} [r(\hat{T}) - r(x)]a(x)\, dx = 1 - \int_0^{\hat{T}} e^{-\delta x}F'(x)\, dx + r(\hat{T})A(\hat{T}) > 0, \quad \text{so} \quad \frac{\partial \hat{T}}{\partial C_2} > 0,$$

the condition IFR implies $\partial\hat{T}/\partial C_1 \leq 0$, and g_0 increasing implies $\partial\hat{T}/\partial C_3 \leq 0$. To see that $\partial\hat{T}/\partial\delta > 0$, write

(8)
$$0 < \int_0^{\hat{T}} [H(x) - H(\hat{T})]A(x)\,dx$$
$$= \int_0^{\hat{T}} \left[\left(\int_0^x \varphi(u)a(u)\,du + C_2\right)/A(x) - \varphi(\hat{T})\right]A(x)\,dx$$
$$= \int_0^{\hat{T}} \left(\int_0^x (\varphi(u) - \varphi(\hat{T}))a(u)\,du + C_2\right)dx$$
$$= \int_0^{\hat{T}} \left(\int_x^{\hat{T}} (\varphi(\hat{T}) - \varphi(u))a(u)\,du\right)dx$$
$$= \int_0^{\hat{T}} x[\varphi(\hat{T}) - \varphi(x)]a(x)\,dx.$$

Keeping F, g, C_1 and C_2 fixed and concentrating on the dependence of $\hat{T}$ on δ, writing $\hat{T} = \hat{T}(\delta)$, the question is how $\hat{T} : [0, \infty) \to (0, \infty]$ behaves generally. (The minimum problem may have several solutions, but for definiteness we then take $\hat{T}$ as the largest of these.) For φ increasing, the integral determining ψ decreases as δ increases, so $\hat{T} = \inf \{T: \psi(T) > 0\}$ cannot decrease:

Proposition 4. Assume φ increasing. Then $\hat{T}$ increases with δ. Further, in the general case we have, since $\lim_{\delta\to\infty} \psi(T) = -C_2$ uniformly on $\{0 \le T \le K\}$ for every $K > 0$, $\lim_{\delta\to\infty} \hat{T}(\delta) = \infty$.

Generally, however, $\hat{T}$ is not increasing. It may jump downwards as well as upwards at discontinuity points where the minimum problem has several solutions. Roughly, this is the case: If at $\delta = \delta_1$ we have two global minima $T_1 < T_2$, then the more φ between T_1 and T_2 is large close to T_1 and small elsewhere, the more likely $\hat{T}$ is to jump downwards from T_2 to T_1 at $\delta = \delta_1$. This can be realized from the following considerations. Assume $\varphi'(T_i) > 0$, $i = 1, 2$. Write $T_i(\delta)$ for the functions, continuous and differentiable at δ_1, giving local minimum points $(T_i = T_i(\delta_1))$. Then

$$\frac{d}{d\delta} H(T_1(\delta), \delta)_{\delta=\delta_1} > \frac{d}{d\delta} H(T_2(\delta), \delta)_{\delta=\delta_1}$$

implies an upwards jump for $\hat{T}$ and vice versa. Now $(i = 1, 2)$

(9)
$$\frac{d}{d\delta} H(T_i(\delta), \delta)_{\delta=\delta_1} = \frac{d}{d\delta} \varphi(T_i(\delta))_{\delta=\delta_1}$$
$$= \varphi'(T_i(\delta_1))T_i'(\delta_1)$$
$$= \frac{\int_0^{T_i} x[\varphi(T_i) - \varphi(x)]a(x)\,dx}{A(T_i)}$$
$$= \frac{\int_0^{T_i} [H(x) - H(T_i)]A(x)\,dx}{A(T_i)}.$$

Now, in order to get a jump downwards, we would like to have H as large as possible between T_1 and T_2. We have, for $T_1 < T < T_2$,

$$H(T)A(T) = \int_0^T \varphi(x)a(x)\,dx + C_2 \le \int_0^{T_2} \varphi(x)a(x)\,dx + C_2 = H(T_2)A(T_2) = H(T_1)A(T_2),$$

or

(10)
$$H(T) \le \frac{H(T_1)A(T_2)}{A(T)}, \qquad T_1 < T < T_2,$$

with equality if $\varphi(x) = 0$ for $T < x < T_2$. It is clear that making g large in $(T_1, T_1 + \epsilon)$ and close to 0 in $(T_1 + \epsilon, T_2)$, for a small positive ϵ, makes $(d/d\delta)H(T_2(\delta), \delta)_{\delta=\delta_1}$ large. This is not transparent for r, since the denominator $A(T)$ in (10) depends on r. In any case, see examples D and E. We have exaggerated example D in order to make the point. Example C exhibits an upward jump.

5. Examples

We present five numerically solved examples, in each case holding F, g, C_1 and C_2 fixed. In A, B and C, $\hat{T}$ solves $\psi(T) = 0$, while in D and E φ has discontinuities giving local minima.

For the special case exponential unit life distribution, $F(x) = 1 - e^{-\lambda x}$, we can write H so that we see that $\hat{T}$ is independent of C_1 and depends on λ and δ only through $\lambda + \delta$:

(11)
$$H(T) = (C_1 - C_2)\lambda + (\lambda + \delta)\frac{\int_0^T g(x)e^{-(\lambda+\delta)x}\,dx + C_2}{1 - e^{-(\lambda+\delta)T}}.$$

A. *Exponential life distribution, linear maintenance cost intensity.*

$F(x) = 1 - e^{-0.1x}$, $g(x) = 10x$, C_1 arbitrary, $C_2 = 180$

This gives

$$\varphi(x) = (C_1 - 180)\cdot 0.1 + 10x;$$

$$\psi(T) = \frac{10}{0.1 + \delta}T + \frac{10}{(0.1 + \delta)^2}e^{-(0.1+\delta)T} - \frac{10}{(0.1 + \delta)^2} - 180.$$

Here φ increases strictly to ∞, so there is exactly one solution to $\psi(T) = 0$ (Case 1). This is Scheaffer's example 1 rescaled.

δ	0	0.02	0.04	0.06	0.08	0.10
$\hat{T}$	6.66	6.81	6.97	7.13	7.30	7.48

By (11), the solution 7.1 for $\delta = 0.06$ holds for example also for $\lambda = 0.15$, $\delta = 0.01$. If we take $g_0(x) = x$, $C_3 = 10$ and put $b = \delta + 0.1$, then, using (7) and the relation $e^{-b\hat{T}} = 1 + 18b^2 - b\hat{T}$, we get

$$\Delta\hat{T} \approx \frac{0.1b\Delta C_2 - 1.8b\Delta C_3 + (36 + 18b\hat{T} - \hat{T}^2)\Delta\delta}{b\hat{T} - 18b^2}.$$

For example when $\delta = 0.06$

$$\Delta\hat{T} \approx 0.024\Delta C_2 - 0.424\Delta C_3 + 8.379\Delta\delta.$$

B. *Rayleigh life distribution, linear maintenance cost intensity.*

$$F(x) = 1 - e^{-(\pi/400)x^2}, \qquad g(x) = 10x,$$
$$C_1 = 300, \qquad C_2 = 180.$$

We have here changed the life distribution of A to one with the same expectation 10 and a linear failure rate.

$$\varphi(x) = 0.1(100 + 6\pi)x;$$

$$\psi(T) = 400(100 + 6\pi)[0.005T + \delta\pi^{-1}] \cdot \left[\Phi\left(0.05\sqrt{2\pi}\,T + \frac{20\delta}{\sqrt{2\pi}}\right) - \Phi\left(\frac{20\delta}{\sqrt{2\pi}}\right)\right] \cdot e^{100\pi^{-1}\delta^2} + 20(100 + 6\pi)\pi^{-1} \cdot e^{-(0.0025\pi T^2 + \delta T)} - 20\pi^{-1}(100 + 15\pi).$$

Here Φ is the $N(0; 1)$ distribution function. Also here $\varphi(x) \uparrow \infty$ strictly, and $\psi(T) = 0$ has a unique solution.

δ	0	0.02	0.04	0.06	0.08	0.10
$\hat{T}$	5.62	5.72	5.83	5.95	6.07	6.20

C. *Exponential life distribution, oscillating maintenance cost intensity.*

$F(x) = 1 - e^{-0.1x}$, $g(x) = \pi x + \cos(2\pi x)$, C_1 arbitrary, $C_2 = 45$.
$\varphi(x) = (C_1 - 45)\cdot 0.1 + \pi x + \cos(2\pi x) \to \infty$, so $\hat{T} < \infty$.

With $b = \delta + 0.1$ we have

$$\psi(T) = \pi b^{-1}T + b^{-1}\cos(2\pi T) + \pi b^{-2}e^{-bT} - (b^2 + 4\pi^2)^{-1}e^{-bT}[2\pi\sin(2\pi T) + 4\pi^2 b^{-1}\cos(2\pi T)] - b(b^2 + 4\pi^2)^{-1} - \pi b^{-2} - 45.$$

δ	0	0.02	0.04	0.06	0.07	0.08	0.10
$\hat{T}$	5.79	5.83	5.88	5.94	6.58	6.61	6.66

With

$$E_n = \left(n - \frac{7}{12}, n + \frac{1}{12}\right),$$

we have $\hat{T} \in \bigcup_{n=0}^{\infty} E_n$, the set of increase for g (Proposition 3). Now $\hat{T} \in E_6$ for $\delta = 0.06$, but $\hat{T} \in E_7$ for $\delta = 0.07$, so that $\hat{T}$ has an upward jump in (0.06, 0.07) as a function of δ. For $\delta = 0.06$ we have two local minima, the first of which is global, while for $\delta = 0.07$ we have two local minima, the second of which is global. This example illustrates the discussion concluding sec. 4, as do the following ones.

D. *The failure rate a simple function, no maintenance cost.*

$$F(x) = \begin{cases} 0, & 0 \le x \le 1 \\ 1 - e^{-100(x-1)}, & 1 \le x \le 1.01 \\ 1 - e^{-1}, & 1.01 \le x \le 37 \\ 1 - e^{-10x+369}, & 37 \le x \end{cases}$$

$g(x) = 0$, $C_1 = 11$, $C_2 = 1$. This is an example of the original model of Barlow & Proschan, Denardo & Fox, since $g(x) = 0$. The failure rate is the simple function

$$r(x) = 100\chi_{(1,1.01)}(x) + 10\chi_{(37,\infty)}(x).$$

Here $\chi_A(x)$ is defined as 1 for x in A and 0 for x not in A. We have local minima in 1 and 37.

$$H(1) = \frac{\delta}{1 - e^{-\delta}},$$

$$H(37) = \frac{1000\dfrac{e^{-\delta} - e^{-1.01\delta-1}}{\delta + 100} + 1}{\dfrac{1 - e^{-\delta}}{\delta} + \dfrac{e^{-\delta} - e^{-1.01\delta-1}}{\delta + 100} + \dfrac{e^{-1.01\delta-1} - e^{-37\delta-1}}{\delta}}.$$

Comparing $H(1)$ and $H(37)$ we get $\hat{T}$.

δ	0	0.02	0.04	0.06	0.08	0.10
$\hat{T}$	37	37	37	1	1	1

E. *Exponential life distribution, g simple.*

$F(x) = 1 - e^{-0.2x}$, $g(x) = 5\chi_{(1,1.5)}(x) + 2\chi_{(4,\infty)}(x)$, C_1 arbitrary, $C_2 = 1$.
Local minima in 1 and 4. With $b = 0.2 + \delta$ we have

$$H(1) = (C_1 - 1)\cdot 0.2 + \frac{b}{1 - e^{-b}},$$

$$H(4) = (C_1 - 1)\cdot 0.2 + \frac{5(e^{-b} - e^{-1.5b}) + b}{1 - e^{-4b}}.$$

δ	0	0.02	0.04	0.06	0.08	0.10
$\hat{T}$	4	4	4	4	1	1

It is easy to see that the condition of Proposition 2 is satisfied for the δ-values tabulated in examples D and E with T_0 equal to 37 and 4, respectively.

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