J. Appl. Prob. **18**, 747–751 (1981)
Printed in Israel
0021–9002/81/030747–05 $00.75

MARKOV CHAINS IN SMALL TIME INTERVALS

STIG I. ROSENLUND,* *University of Göteborg*

Abstract

For a time-homogeneous continuous-parameter Markov chain we show that as $t \to 0$ the transition probability $p_{n,j}(t)$ is at least of order $t^{r(n,j)}$, where $r(n,j)$ is the minimum number of jumps needed for the chain to pass from n to j. If the intensities of passage are bounded over the set of states which can be reached from n via fewer than $r(n,j)$ jumps, this is the exact order.

BIRTH-DEATH PROCESS; LOCAL BEHAVIOUR; MARKOV CHAIN

1. Definitions and results

Let $N(t)$, $t \geqq 0$, be a time-homogeneous Markov chain with the integers as state space. For $t \geqq 0$ let

$$p_{n,j}(t) = P(N(u+t) = j \mid N(u) = n),$$

$$q_{n,j} = p'_{n,j}(0),$$

$$q_n = \sum_{j \neq n} q_{n,j}.$$

We assume $0 \leqq q_n < \infty$ for all n.

The expressions appearing in the theorem below are defined in the following way. Put $r(n,n) = 0$ and $c_{n,n} = 1$. For $n \neq j$ define the subset of the positive integers

$$G_{n,j} = \left\{ r : \exists k_i,\ 0 \leqq i \leqq r, \text{ such that } k_0 = n,\ k_r = j \text{ and } \prod_{i=1}^{r} q_{k_{i-1},k_i} > 0 \right\},$$

which consists of those integers r such that $N(t)$ can pass from n to j via r jumps. If $G_{n,j}$ is empty let $r(n,j) = \infty$ and $c_{n,j} = 0$. If $G_{n,j}$ is not empty, let $r(n,j) = \min G_{n,j}$, which is the minimum number of jumps required to pass from n to j. Let $s^m_{n,j}$, $m = 1, 2, \cdots$, be an enumeration of those sequences $(k_0, \cdots, k_r)$ such that $r = r(n,j)$, $k_0 = n$, $k_r = j$ and $q_{k_0,k_1} \cdots q_{k_{r-1},k_r} > 0$. Here $s^m_{n,j} \neq s^u_{n,j}$ for $m \neq u$, but they may be permutations of each other. Let

Received 7 May 1980; revision received 12 August 1980.
* Postal address: Västmannagatan 93, S-113 43 Stockholm, Sweden.

$$a^m_{n,j} = \prod_{i=1}^{r} q_{k_{i-1},k_i}, \quad \text{with } (k_0, \cdots, k_r) = s^m_{n,j}.$$

Then we can define $c_{n,j} = \Sigma_m a^m_{n,j}$, the sum of $r(n,j)$-long products of intensities of transition starting at n and ending at j. Possibly $c_{n,j} = \infty$. When $q_{n,j} > 0$ we have of course $r(n,j) = 1$ and $c_{n,j} = q_{n,j}$. If $c_{n,j} = 0$, then j is inaccessible from n, so that $p_{n,j}(t) = 0$.

The following result is a generalization of the infinitesimal conditions of the process. Let n and j be fixed integers.

Theorem.

(i) $\liminf_{t\to 0} p_{n,j}(t)t^{-r(n,j)} \geqq c_{n,j}/r(n,j)!$.

(ii) If $q_i \leqq b_{n,j} < \infty$ when $r(n,i) \leqq r(n,j) - 1$, then $c_{n,j} < \infty$ and $p_{n,j}(t) = c_{n,j}t^{r(n,j)}/r(n,j)! + o(t^{r(n,j)})$ as $t \to 0$.

(iii) If $q_i \leqq b_{n,j} < \infty$ when $r(n,i) \leqq r(n,j)$, then the residual in (ii) is $O(t^{r(n,j)+1})$.

In words, $p_{n,j}(t)$ is always at least of order $t^{r(n,j)}$. If q_i is bounded over the set of states i which can be reached from n via fewer than $r(n,j)$ jumps, this is the exact order. If, in addition, q_i is bounded over the set of states accessible from n in $r(n,j)$ jumps, the remainder term is at most of order $t^{r(n,j)+1}$. We note that the same result holds for the distribution function $F_{n,j}$ for a first-passage time from n to j, since $F_{n,j}$ is equal to $p_{n,j}$ when $q_j = 0$ and $c_{n,j}$ does not depend on the $q_{j,i}$.

The assumption of (ii) holds if at each state i such that $r(n,i) \leqq r(n,j) - 2$ there are only finitely many states that can be reached in a single jump, because then the set of states i of the assumption is finite. Similarly for the assumption of (iii), which thus is true for example for a birth–death process with birth rates $\lambda_n = q_{n,n+1}$ and death rates $\mu_n = q_{n,n-1}$. Here $r(n,j) = |n - j|$ and

$$c_{n,j} = \begin{cases} \mu_{j+1} \cdots \mu_n, & j < n \\ \lambda_n \cdots \lambda_{j-1}, & j > n. \end{cases}$$

For a finite Markov chain the result of (iii) follows from the representation

$$P(t) = e^{tQ} = I + tQ + (t^2/2)Q^2 + (t^3/6)Q^3 + \cdots,$$

where $P(t) = \{p_{n,j}(t)\}$ and $Q = \{q_{n,j}\}$, if one observes that $\{Q^k\}_{n,j}$ is 0 for $k < r(n,j)$ and $c_{n,j}$ for $k = r(n,j)$. Besides its intrinsic interest the theorem is, for instance, useful as a check on the leading term of a MacLaurin expansion for $p_{n,j}$, or as a check on the order of its Laplace transform at ∞. In applied probability the theorem can be used in problems of Markovian decision processes which are Markov chains controlled at the discrete set of time epochs $h, 2h, 3h, \cdots$. The limiting behaviour of objective functions as $h \to 0$ can be obtained with the aid of the theorem. See Rosenlund (1978b), p. 141.

2. Proof

If $n = j$ or $c_{n,j} = 0$ the theorem is trivially true. Thus let n and j be fixed such that $n \neq j$ and $c_{n,j} > 0$. Assume that $N(0) = n$. Let $r(n,j) = r$ and $s^m_{n,j} = (k_{m,0}, \cdots, k_{m,r})$. Define the event

$A_{m,t} = \{N(t) = j$ and the passage from n to j goes over $s^m_{n,j}$ with exactly r jumps$\}$.

Let T_1 be the time of the first jump and let T_k be the length of time between the $(k-1)$th and the kth jump. If $N(t) = j$ then either some $A_{m,t}$ has occurred or else more than r jumps have taken place in $(0, t]$, i.e.

$$\left\{ \bigcup_m A_{m,t} \right\} \subset \{N(t) = j\} \subset \left\{ \bigcup_m A_{m,t} \right\} \cup \{T_1 + \cdots + T_{r+1} \leqq t\}.$$

Now $A_{m,t}$ occurs if and only if the following takes place. First $T_1 = t_1 < t$ and the first jump is to $k_{m,1}$. The density of T_1 is $q_n \exp(-q_n t_1)$ and the probability of a jump to $k_{m,1}$ is $q_{n,k_{m,1}}/q_n$. Secondly $T_2 = t_2 < t - t_1$ and the second jump is to $k_{m,2}$. Given $N(T_1) = k_{m,1}$ the density of T_2 is $q_{k_{m,1}} \exp(-q_{k_{m,1}} t_2)$ and the probability of a jump to $k_{m,2}$ is $q_{k_{m,1},k_{m,2}}/q_{k_{m,1}}$. We can go on in this way until $N(T_1 + \cdots + T_r) = j$. Then finally T_{r+1} must be greater than $t - t_1 - \cdots - t_r$ and this has the conditional probability $\exp(-q_j(t - t_1 - \cdots - t_r))$. In other words

$$P(A_{m,t}) = \int_0^t q_{n,k_{m,1}} \exp(-q_n t_1) \int_0^{t-t_1} q_{k_{m,1},k_{m,2}} \exp(-q_{k_{m,1}} t_2)$$

$$\cdots \int_0^{t-t_1-\cdots-t_{r-1}} q_{k_{m,r-1},j} \exp(-q_{k_{m,r-1}} t_r) \exp(-q_j(t - t_1 - \cdots - t_r)) dt_r \cdots dt_1.$$

With

$$b_m = \max(q_n, q_{k_{m,1}}, \cdots, q_{k_{m,r-1}}, q_j)$$

we thus have

$$\sum_m \exp(-b_m t) a^m_{n,j} t^r/r! \leqq p_{n,j}(t) \leqq \sum_m a^m_{n,j} t^r/r! + P(T_1 + \cdots + T_{r+1} \leqq t).$$

From the left inequality we get (i) by monotone convergence. Under the assumption of (ii) it holds $b_m \leqq \max(b_{n,j}, q_j) = b$, say. Furthermore, by conditioning with respect to $N(T_1), \cdots, N(T_1 + \cdots + T_r)$ it is easy to show that $P(T_1 + \cdots + T_{r+1} \leqq t) \leqq P(T_{r+1} \leqq t) F_b^{r*}(t)$, where $F_b(t) = 1 - e^{-bt}$. Hence

$$e^{-bt} c_{n,j} t^r/r! \leqq p_{n,j}(t) \leqq c_{n,j} t^r/r! + P(T_{r+1} \leqq t) \sum_{k=r}^{\infty} e^{-bt} (bt)^k/k!.$$

This proves (ii). Finally, if the assumption of (iii) holds, then

$$P(T_1 + \cdots + T_{r+1} \leqq t) \leqq F_b^{(r+1)*}(t) = O(t^{r+1}).$$

3. Example

We shall illustrate that the conditions of the theorem cannot be much weakened. Let $q_{-2} = 0$, $q_{-1} = q_{-1,-2} = 1$, $q_0 = 2$, $q_{0,-1} = 2 - \pi^2/6 = a \approx 0.355$, $q_{0,k} = k^{-2}$ for $k \geqq 1$, $q_k = q_{k,-k-2} > 0$ for $k \geqq 1$, $q_k = q_{k,-2} = (k+2)^2$ for $k \leqq -3$. We shall study $p_{0,-2}(t)$ for the cases $q_k = 1$ and $q_k = 2k^2$ $(k \geqq 1)$. For the first case the condition of (ii) is satisfied, but not that of (iii), and the conclusion of (iii) does not obtain for $n = 0$ and $j = -2$. For the second case the condition of (ii) is not satisfied, and the conclusion of (ii) does not obtain. We have $r(0, -2) = 2$ and $c_{0,-2} = a$. Assume $N(0) = 0$. Let

$A_t = \{N(t) = -2$ and the passage goes via -1 in two jumps$\}$,

$B_{k,t} = \{N(t) = -2$ and the passage goes via k and $-k-2$ in three jumps$\}$.

We have

$$P(A_t) = \int_0^t a \exp(-2t_1) \int_0^{t-t_1} \exp(-t_2) dt_2 dt_1$$

$$= \tfrac{1}{2}a - ae^{-t} + \tfrac{1}{2}ae^{-2t} = \tfrac{1}{2}at^2 + O(t^3),$$

$$P(B_{k,t}) = \int_0^t k^{-2} \exp(-2t_1) \int_0^{t-t_1} q_k \exp(-q_k t_2) \int_0^{t-t_1-t_2} k^2 \exp(-k^2 t_3) dt_3 dt_2 dt_1.$$

Let $H(t) = \sum_{k=1}^{\infty} P(B_{k,t})$, then $p_{0,-2}(t) = P(A_t) + H(t)$. Now $12\pi^{-2}H(t)$ is a distribution function, and the Laplace transform $\hat{H}(s) = \int_0^\infty e^{-st} dH(t)$ is

$$\hat{H}(s) = \sum_{k=1}^{\infty} \frac{q_k}{(s+2)(s+q_k)(s+k^2)}\,.$$

When $q_k = 1$ we get

$$\hat{H}(s) = \{2s(s+1)(s+2)\}^{-1}\{\pi s^{\frac{1}{2}}(e^{2\pi s^{\frac{1}{2}}} + 1)/(e^{2\pi s^{\frac{1}{2}}} - 1) - 1\} \sim \tfrac{1}{2}\pi s^{-2.5} (s \to \infty).$$

Hence $H(t) \sim (4\pi^{\frac{1}{2}}/15)t^{2.5}$ $(t \to 0)$ (see Feller (1971), p. 632 and p. 445). It follows that $p_{0,-2}(t) = \frac{1}{2}at^2 + O(t^{2.5})$, but the conclusion of (iii) does not hold. When $q_k = 2k^2$ we have

$$\hat{H}(s) = \frac{2}{s+2} \sum_{k=1}^{\infty} \left(\frac{1}{s+k^2} - \frac{1}{s+2k^2} \right)$$

$$= \frac{1}{s(s+2)} \left(\frac{\pi s^{\frac{1}{2}}(e^{2\pi s^{\frac{1}{2}}} + 1)}{e^{2\pi s^{\frac{1}{2}}} - 1} - \frac{\pi (\frac{1}{2}s)^{\frac{1}{2}}(e^{2\pi(\frac{1}{2}s)^{\frac{1}{2}}} + 1)}{e^{2\pi(\frac{1}{2}s)^{\frac{1}{2}}} - 1} \right)$$

$$\sim \pi\left(1 - \frac{1}{2^{\frac{1}{2}}}\right)s^{-1.5} \qquad (s \to \infty).$$

Hence

$$p_{0,-2}(t) \sim H(t) \sim \tfrac{4}{3}\left(1 - \frac{1}{2^{\frac{1}{2}}}\right)\pi^{\frac{1}{2}} t^{1.5} \qquad (t \to 0).$$

It is easily seen that this counterexample to putative generalizations of the theorem will hold even if we make the chain recurrent by allowing transitions from -2.

References

FELLER, W. (1971) *An Introduction to Probability Theory and its Applications,* Vol. 2, 2nd edn. Wiley, New York.

ROSENLUND, S. I. (1978a) Transition probabilities for a truncated birth–death process. *Scand. J. Statist.* **5**, 119–122.

ROSENLUND, S. I. (1978b) Controlling a Markovian queue at equidistant time epochs. *Trans. 8th Prague Conf. Information Theory, Statistical Decision Functions, Random Processes,* Vol. B, ed. M. Driml. Academia, Prague, 129–143.