Inference in multiplicative pricing
Lecture in Indonesia november 2014

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I will give a review of the paper by me with this title from 2014 in

Scandinavian Actuarial Journal **2014**(8), 690-713.
http://www.tandfonline.com/doi/abs/10.1080/03461238.2012.760885.

Please interrupt me with questions at any time!

Descriptions of tariff analysis point estimate methods

MMT – Method of Marginal Totals

<u>Model</u>: Risk premium is multiplicative in the arguments.

<u>Method</u>: Solve a system of equations defined by prescribing that the sum of multiplicatively computed estimated claim costs over any argu-

ment class - any marginal - be equal to the empirical claim cost of the argument class.

Standard GLM

Model: Claim numbers are Poisson distributed. Claim severities are Γ distributed with a constant CV (coefficient of variation).

Method: Solve the ML (maximum likelihood) equations resulting from the model.

Tweedie for risk premium

Model: Claim costs have variance function $v(\mu) = \mu^p$ with $p \in [1, 2]$.

Method: Solve the ML equations resulting from the model.

Tweedie with exponent $p = 1$ gives the MMT solution. The case $p = 2$ was historically denoted the "direct" method.

Descriptions of tariff analysis variance estimate methods

MVW – MMT Variance estimates under Weak assumptions

Model: Claim frequency and mean claim are multiplicative in the arguments. The claim cost of a tariff cell is distributed Compound Poisson.

Method: The GLM Poisson log link model for claim numbers is used for claim frequency. For mean claim I take the estimated CVs for univariate mean claims as a start values. These are adjusted upwards by factors resembling the ratios of GLM claim frequency variance estimates to simple univariate claim frequency variance estimates.

Standard GLM and Tweedie

Models: As before.

Methods: The variance estimates that follow from the GLM theory.

The following are my main conclusions.

1. With sufficiently many claims *or* sufficiently many arguments MMT is preferable over Standard GLM in tariff analysis.

2. The MVW confidence interval method for MMT is mostly preferable to Standard GLM.

3. The Tweedie model for risk premium should not be used.

To show or disprove conclusion **1**, I had to define preferable by means of a measure of goodness-of-fit for a method. I proposed the exposure-weighted mean square deviation of estimated risk premium from true risk premium, summed over all tariff cells. Thus, let u be the index of a tariff cell $\in \{1, 2, \ldots, n\}$ and

$$\begin{aligned} e_u &= \text{exposure in cell } u \\ \tau_u &= \text{true risk premium for cell } u \\ \widehat{\tau}_u^{(\mathrm{X})} &= \text{estimate of } \tau_u \text{ for a method X} \end{aligned}$$

and define the goodness-of-fit measure

$$\mathcal{M}(\mathrm{X}, \{e_u\}) = \mathrm{E}\left[\sum_{u=1}^{n} e_u \left(\widehat{\tau}_u^{(\mathrm{X})} - \tau_u\right)^2\right] / \sum_{u=1}^{n} e_u$$

Now introduce a volume measure c such that

$$e_u = ce_u^0$$

for some suitably normed sequence $\{e_u^0\}$. The interest is in how the goodness-of-fit measure for X, compared to some other method, behaves for different values of c.

Thus I defined

$$\begin{aligned} \mathcal{M}_M(c) &= \mathcal{M}(\text{MMT}, \{ce_u^0\}) \\ \mathcal{M}_S(c) &= \mathcal{M}(\text{Standard GLM}, \{ce_u^0\}) \end{aligned}$$

Consider a random variable Z with mean μ that we want to be close to a number a. It holds

$$\mathrm{E}[(Z - a)^2] = \mathrm{Var}[Z] + (a - \mu)^2$$

or in words

$$\text{Mean square error} = \text{Variance} + \text{bias}^2$$

We can generalize this to the collection of all tariff cells.

I and other authors have found indications that normally, vaguely speaking, Standard GLM has smaller variance than MMT. Both methods have biases. I conjectured in my article that variances, suitably defined for the collection of all tariff cells, k_1/c for MMT and k_2/c for S-GLM, where $k_1 > k_2$, hold asymptotically for both $c \to 0$ and $c \to \infty$, albeit possibly with one pair (k_1, k_2) for $c \to 0$ and another pair (k_1, k_2) for $c \to \infty$. This conjecture was corroborated by my simulations. The means that when $c <$ some constant, then Standard GLM will be preferable.

For the biases, I had to distinguish between

A. The true risk premiums are exactly multiplicative.

B. The true risk premiums deviate from exact multiplicativity.

In case A both methods will have asymptotic zero bias as $c \to \infty$. If my conjecture is correct, then Standard GLM will be preferable both as $c \to 0$ and as $c \to \infty$.

On the other hand, if B holds, which almost always will be the case for more than one argument, then the comparison of asymptotic goodness-of-fit for the two methods will be determined by the now non-zero asymptotic biases, suitably defined.

I conjectured that in case B the asymptotic bias of MMT is typically smaller than the asymptotic bias of Standard GLM. This was also corroborated by my simulated cases. These biases could be determined exactly by taking cases were all observed tariff risk premiums were exactly equal to their expected values.

Thus, since the variances tend to 0, I found for my cases

$$\lim_{c\to\infty} \mathcal{M}_M(c) < \lim_{c\to\infty} \mathcal{M}_S(c)$$

This inequality is not, however, universally true. I have made calculations for the simple 2×2 case (two arguments with two classes each). In one of about 10 cases I found the reverse inequality. This case had multiplicative mean claim, which speaks for Standard GLM, and non-multiplicative claim frequency.

Apart from these exceptions, there should normally be an indifference value c_0 of c such that

- ◇ $\mathcal{M}_M(c) > \mathcal{M}_S(c)$ for $c < c_0$ — Standard GLM is preferable.
- ◇ $\mathcal{M}_M(c_0) = \mathcal{M}_S(c_0)$ — Standard GLM and MMT are equal.
- ◇ $\mathcal{M}_M(c) < \mathcal{M}_S(c)$ for $c > c_0$ — MMT is preferable.

In words, for sufficiently small exposure with few claims Standard GLM is preferable. For sufficiently large exposure with many claims MMT is preferable. That is the first part of my conclusion **1**. The second part is that sufficiently many arguments make MMT preferable. This I demonstrated in my simulations, where I considered a sequence of cases where more and more arguments were made non-multiplicative. Then the indifference value c_0 decreased more and more.

The way I made the arguments non-multiplicative was by letting their factor ladders depend on some other argument. Another way to construct non-multiplicative risk premium is to set

$$\text{risk premium} = \alpha(\text{multiplicative risk premium}) + (1-\alpha)\text{constant}.$$

I have considered such cases in unpublished research. I saw the same phenomenon with an indifference value c_0 of c such that Standard GLM is preferable below it and MMT is preferable above it.

Before my article was published there was some debate among actuaries and in universities in Sweden about my hypothesis. A student was assigned an examination work by an insurance company (not my employer Länsförsäkringar Alliance) and Stockholm University. She studied a simulated case as that above. She found Standard GLM to be preferable. My interpretation of her study is that she had set the exposure factor c below c_0 and that a larger c would have made MMT preferable.

To show or disprove conclusion **2**, I simulated some cases resembling the ones for showing conclusion **1**. But I made those cases exactly multiplicative in claim frequency and mean claim. I did not want to burden the comparisons by having to ascertain what parameters – different for different methods – I estimated.

I studied 95 % level confidence intervals in the GLM form. I.e. zero width for a base class with factor 1 is applied to make possible a comparison of the methods. Base class 1 was used. Otherwise I normally in practice give confidence intervals with all non-zero widths. The article explains their proper interpretation.

The Standard GLM variance estimates and ensuing confidence intervals were computed with the Pearson χ^2-based dispersion parameter estimate using individual claims. It is the best estimate. The formulas of MVW are given last in the article.

The probability that a confidence interval covers the true value should be close to 0.95. Interval widths should be small. These are of course conflicting goals. I formulated no precise rule for weighting them together, but instead looked at the overall picture of cover probabilities for different classes and the mean widths.

For cases where the claim severity CV was constant I found Standard GLM to be preferable by having smaller mean confidence intervals widths, while the cover probabilities were 0.95. This was as I expected, but that this held even for other claim severity distributions than the Γ-distribution can be said to be a new result corroborating previous conjectures.

When I departed from the constant CV, the results showed MVW to be best. Standard GLM gave too high cover probailities for some classes and too low for other cases, while MVW had approximately 0.95 cover probaility for all classes. This came at a price of somewhat higher mean widths by a factor about 1.05. Imposing stronger distributional assumptions implies smaller widths – good if the assumptions are true.

Since the constant CV assumption is very strong, i.e. not realistic, this shows my conclusion **2**.

Are there then no drawbacks to the MVW method, given that MMT is preferable by the criteria for point estimates? Yes, there are. Firstly, if the exposure is extremely non-multiplicative we can get some wrong confidence intervals. Secondly, MVW depends on there being enough claims in a class of an argument to give a good confidence interval for the class. Classes with too few claims will normally get high intervals widths anyway, but it might happen that an interval width is misleadingly low. It is a form of overparametrization. Such confidence intervals will have to be adjusted manually.

Conclusion **3** that the Tweedie risk premium method should not be used: For point estimates the value $p = 1$ gives the MMT solution. In my simulated cases the value $p = 1.5$ gave the Standard GLM solution. Values other than these were of no use. It was never better than S-GLM and necessitates additional work to estimate the exponent in practice. For confidence intervals I also found that some value of p gave the Standard GLM solution, while other values gave worse solutions. So the same conclusion holds.

Other methods

There are generalizations of Standard GLM such that both risk premiums and dispersion parameters vary over tariff cells and obey GLM models. The log link gives multiplicative risk premiums and dispersion parameters. The claim severity CV is a function of the dispersion parameter, so it is not constant. See for example

Smyth, G. K. & Jørgensen, B. (2002), Fitting Tweedie's Compound Poisson Model to Insurance Claims Data: Dispersion Modelling. *ASTIN Bulletin* **32(1)**, 143–157.

In my opinion these generalizations are not much better, since they impose the strong assumption of dispersion parameter multiplicativity.