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Upwards Passage Times in the Non-negative Birth-death Process

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ABSTRACT. The birth–death process on the non-negative integers is studied with respect to the first passage time from n to j with $j>n$. The associated density $f_{n,j}$ is a sum of j real exponential terms. Simple direct proofs are given for Keilson's (1971) results that $f_{0,j}$ is a convolution of j distinct exponential densities and that $f_{n,n+1}$ is a mixture of $n+1$ exponential densities, using a result by Ledermann & Reuter (1954). For a general birth–death queue the passage time analysis yields the distribution of the LIFO waiting time; conditional on being positive this is distributed according to a mixture of exponential densities.

Key words: birth–death process, LIFO waiting time, passage time

1. Main result

Consider the birth–death process $N(t)$ on the state space of non-negative integers, governed by rates λ_n for transition from state n to state $n+1$ and μ_n for transition from state n to $n-1$ with $\lambda_n>0$, $n=0$, 1, 2, ...; $\mu_n>0$, $n=1, 2, \ldots$; and $\mu_0=0$. Let $f_{n,j}$ be the density of the first passage time from n to j and $\sigma_{n,j}$ its Laplace transform;

$$f_{n,j}(x)=\frac{d}{dx}P(\inf\{\tau:\tau>0,\ N(t+\tau)=j\}\leqslant x\,|\,N(t)=n),$$

$$\sigma_{n,j}(s)=\int_0^\infty e^{-sx}f_{n,j}(x)\,dx.$$

Then from Keilson (1965, p. 407) we have, defining

$$\sigma_n^+(s)=\sigma_{n,n+1}(s),$$

$$\begin{cases}\sigma_0^+(s)=\lambda_0(s+\lambda_0)^{-1}\\ \sigma_n^+(s)=\lambda_n(s+\lambda_n+\mu_n-\mu_n\sigma_{n-1}^+(s))^{-1}, \quad n=1, 2, \ldots;\end{cases} \tag{1}$$

This relation can be realized from a simple consistency argument, using the strong Markov property. It permits us to calculate all $\sigma_{n,j}$ with $j>n$, since by the strong Markov property

$$\begin{cases} f_{n,j}=f_{n,n+1}*f_{n+1,n+2}*\ldots*f_{j-1,j},\\ \sigma_{n,j}(s)=\sigma_n^+(s)\sigma_{n+1}^+(s)\ldots\sigma_{j-1}^+(s).\end{cases} \tag{2}$$

The first step in the inversion of $\sigma_{n,j}$ is to write it as a rational function. Define the polynomials P_N of degree N

$$\begin{cases} P_0(s)=1,\ P_1(s)=s+\lambda_0,\\ P_N(s)=(s+\lambda_{N-1}+\mu_{N-1})P_{N-1}(s)-\lambda_{N-2}\mu_{N-1}P_{N-2}(s),\\ \qquad N>1.\end{cases} \tag{3}$$

Confer Keilson (1971, p. 392). We use identical notation as this author, except that $S_{n,j}$ is here denoted $f_{n,j}$. By straightforward induction we obtain

$$\sigma_n^+(s)=\lambda_nP_n(s)/P_{n+1}(s),\ n=0, 1, \ldots. \tag{4}$$

Inserting (4) in (2) we get immediately

$$\sigma_{n,j}(s)=\lambda_n\lambda_{n+1}\ldots\lambda_{j-1}P_n(s)/P_j(s),\ j>n. \tag{5}$$

Keilson (1964) gives a similar expression for $\sigma_{n,j}$, eq. (5.13), obtained via the Laplace transform of the transition probability function of the process. It does not seem clear of what degrees the polynomials involved are. A representation of $f_{n,j}$ as a finite sum of exponential terms, eq. (5.14), is further obtained via a spectral representation of the transition probability function due to Ledermann & Reuter (1954). The procedure is however much simplified by a direct application of Lemma 1, p. 328, in Ledermann & Reuter (1954) to the passage time problem. We give here a proof of (6) below along the lines of these authors. Let $-\theta_{N,1}, \ldots, -\theta_{N,N}$ be the zeros of P_N, ordered so that $\mathrm{Re}\,(\theta_{N,1})\leqslant\mathrm{Re}\,(\theta_{N,2})\leqslant \ldots \leqslant\mathrm{Re}\,(\theta_{N,N})$.

Proposition. *The $\theta_{N,r}$ and $\theta_{N+1,r}$ are distinct, real and satisfy*

$$\theta_{N+1,1}<\theta_{N,1}<\theta_{N+1,2}<\theta_{N,2}<\ldots<\theta_{N,N-1}<\theta_{N+1,N}<\theta_{N,N}<\theta_{N+1,N+1}. \tag{6}$$

It then follows from Feller (1971, pp. 438–439) that $f_{n-1,n}$ has the exponential mixture form (3.13).

Proof. The statement is equivalent to each of the following two statements.

The $\theta_{N,r}$ are distinct and real, and $P_{N+1}(-\theta_{N,r}) \neq 0$ and has the same sign as $(-1)^r$. (6′)

The $\theta_{N+1,r}$ are distinct and real, and $P_N(-\theta_{N+1,r}) \neq 0$ and has the same sign as $(-1)^{r+1}$. (6″)

Let us for example show the equivalence between (6) and (6′). First observe that the coefficient of s^n in P_n is 1. If (6) holds, then P_{N+1} changes sign at each of its zeros, so that $P_{N+1}(-\theta_{N,r})$ changes sign as r increases one step. Further $\lim_{s\to\infty} P_{N+1}(s) = \infty$ implies $\operatorname{sgn}(P_{N+1}(-\theta_{N,1})) = -1$. Hence (6′) holds. Conversely, if (6′) holds, then $P_{N+1}(-\theta_{N,r})$ changes sign as r increases one step, so that P_{N+1} has at least one zero in each of the $N-1$ intervals $(-\theta_{N,r}, -\theta_{N,r-1})$. Moreover $\operatorname{sgn}(P_{N+1}(-\theta_{N,1})) = -1$ and $\lim_{s\to\infty} P_{N+1}(s) = \infty$ implies a zero of P_{N+1} to the right of $-\theta_{N,1}$. Also, $\operatorname{sgn}(P_{N+1}(-\theta_{N,N})) = (-1)^N$ and $\lim_{s\to-\infty} P_{N+1}(s) = (-1)^{N+1}\cdot\infty$ implies a zero to the left of $-\theta_{N,N}$. Hence P_{N+1} has $N+1$ real zeros situated according to (6).

If the statements are true all $\theta_{N,r}$ are positive, because otherwise the Laplace transform (4) would be unbounded for $s>0$, which is impossible.

Now (6′) is true for $N=1$ since $P_2(-\theta_{1,1}) = -\lambda_0\mu_1$ by (3). Assume (6) and hence (6″) for $N=i-1$. Then for $N=i$ by (3) $P_{i+1}(-\theta_{i,r}) = -\lambda_{i-1}\mu_i P_{i-1}(-\theta_{i,r})$, which is not zero and has sign as $(-1)^{1+r+1} = (-1)^r$. Thus (6′) and hence (6) holds for $N=i$. Q.E.D.

We invert (5) for $j>n\geq 0$ and obtain

$$f_{n,j}(x) = \sum_{r=1}^{j} \frac{\lambda_n \cdots \lambda_{j-1} P_n(-\theta_{j,r})}{P_j'(-\theta_{j,r})} e^{-\theta_{j,r}x}, \quad \theta_{j,r}>0. \quad (7)$$

Some asymptotic relations can be derived. For large x the first term for $r=1$ of course preponderates and similarly we have for the moments

$$\int_0^\infty x^r f_{n,j}(x)\,dx \sim \frac{\lambda_n \cdots \lambda_{j-1} P_n(-\theta_{j,1})\, r!}{P_j'(-\theta_{j,1})\,\theta_{j,1}^{r+1}} \quad (r\to\infty). \quad (8)$$

(We know that $P_n(-\theta_{j,1})>0$, since $-\theta_{j,1} > -\theta_{j-1,1} > \ldots > -\theta_{n,1}$.) Applying Theorem 4 in Feller (1971, p. 446) and the assertion of Theorem 3, p. 445, which is valid also for Theorem 4, we further get

$$f_{n,j}(x) \sim \lambda_n \ldots \lambda_{j-1} x^{j-n-1}/(j-n-1)! \quad (x\to 0). \quad (9)$$

Exact expressions for the moments of $f_{n,j}$ in terms of λ_k and μ_k can be obtained from (1): Move the denominator to the left hand side, differentiate in s and put $s=0$. This gives a simple recursive relation for the first moment of $f_{n,n+1}$. Further differentiations give recursive relations for the higher moments. The moments of $f_{n,j}$ are then obtained with the aid of (2). Cf. Keilson (1965). We here give the expectations (define $\prod_{i=r+1}^{r} = 1$)

$$\int_0^\infty x f_{n,j}(x)\,dx = \sum_{r=n}^{j-1} \sum_{k=0}^{r} \lambda_{r-k}^{-1} \prod_{i=r-k+1}^{r} \mu_i \lambda_i^{-1}, \quad j>n. \quad (10)$$

2. Connection with Keilson's results

Setting $n=0$ in (5) we get

$$\sigma_{0,j}(s) = \lambda_0 \ldots \lambda_{j-1}((s+\theta_{j,1}) \ldots (s+\theta_{j,j}))^{-1}, \quad (11)$$

so that with (6) we see that $f_{0,j}$ is a convolution of j distinct exponential densities. Thus we have arrived at the result (4) in Keilson (1971). For $j=n+1$ it follows from (6) that each term of (7) is positive (Feller, 1971, pp. 438–439). This holds only for $j=n+1$, since for $j>n+1$ the degree of P_j differs by more than one from the degree of P_n. The result (11) in Keilson (1971) follows on noting that (7) holds also for a birth–death process truncated at K ($\lambda_K = 0$) provided $j\leq K$. A downwards transition in a process $N(t)$ truncated at K is an upwards transition in the truncated birth–death process $N^*(t) = K - N(t)$ with birth rates $\lambda_n^* = \mu_{K-n}$ and death rates $\mu_n^* = \lambda_{K-n}$. The coefficients θ_j and β_j of (11) in Keilson (1971) will be found by applying the procedure above to $N^*(t)$ to get $f_{K-1,K}^* = f_{1,0} = R$.

Our procedure to establish (6) and the relations (4) and (11) in Keilson (1971) implies thereby is simple and elementary. Keilson, on the other hand, obtained his result (11) by deep methods, using the reversibility in time of the birth–death process and a spectral representation of the transition probability function $p_{00}(t)$.

3. Queuing applications

The busy period and other passage times in birth–death queuing models have recently attracted some attention in the literature, see Conolly (1974), Natvig (1975*a*, *c*) and Rosenlund (1973 and 1975). For the case of constant transition rates, $\lambda_n = \beta (n\geq 0)$ and $\mu_n = \lambda (n\geq 1)$, the polynomials P_N may be expressed in closed forms; put $u=w=1$ in (11), (12) and (13) of Rosenlund (1975). For an m-server birth–death queue such that a customer who joins the system stays there until service completion (i.e. no reneging or push-out), the passage time analysis yields the LIFO waiting time distribution; a customer who finds k customers before him in the system at arrival ($k\geq m$) and does not balk has, under reverse order service, to wait during a first passage time from $k+1$ to k. Let ξ_n be the balking probability for a customer finding n customers before him and let λ be the Poisson arrival intensity; then $\lambda_n = \lambda(1-\xi_n)$ for the process $N(t)$ = number of customers in the system at t. Now assume a finite waiting room size N, i.e. $\lambda_{m+N}=0$, while $\lambda_n>0$ for $0 \leq n < m+N$ and $\mu_n > 0$ for $1 \leq n \leq m+N$. Let π_k be the steady-state probability that a non-balking customer finds k customers before him at arrival. From Natvig (1975*b*, eq. (2.7)) we have

$$\pi_k = a \prod_{i=1}^{k} \lambda_i \mu_i^{-1}, \quad k=0, ..., m+N-1. \tag{12}$$

The density of the waiting time W, conditional on $W>0$, is then

$$w(x) = \frac{d}{dx} P(W \leq x \mid W > 0)$$
$$= \sum_{k=m}^{m+N-1} \pi_k(\pi_m + \dots + \pi_{m+N-1})^{-1} f_{k+1,\,k}(x). \tag{13}$$

Now $f_{k+1,k} = f^*_{m+N-k-1,m+N-k}$, where $f^*_{n,j}$ is the density of a passage time from n to j in $N^*(t) = m+N-N(t)$. Hence $f_{k+1,k}$ is a mixture of $m+N-k$ exponential densities, and so w is a mixture of $N(N+1)/2$ exponential densities.

The expression (13) has to be supplemented with the steady-state probabilities for immediate service commencement, waiting and loss, respectively, in order to characterize the fate of a customer. Put

$$b = \sum_{r=0}^{m+N} \lambda\lambda_0^{-1} \prod_{i=1}^{r} \lambda_{i-1} \mu_i^{-1}.$$

Then

$$\begin{cases} P\,(\text{immediate service commencement}) \\ \qquad = b^{-1} \sum_{r=0}^{m-1} \prod_{i=1}^{r} \lambda_i \mu_i^{-1}; \\ P(W>0) = b^{-1} \sum_{r=m}^{m+N-1} \prod_{i=1}^{r} \lambda_i \mu_i^{-1}; \\ P\,(\text{loss}) = 1 - b^{-1} \sum_{r=0}^{m+N-1} \prod_{i=1}^{r} \lambda_i \mu_i^{-1}. \end{cases} \tag{14}$$

The losses include customers who find the waiting room occupied as well as customers who balk despite available waiting place. For the steady-state distribution of an arriving customer, see e.g. Natvig (1975b), eq. (2.24) and lines 7 and 8 on p. 590. The LIFO waiting time in the queue $M/M/m/N$, i.e. $\lambda_n = \lambda (0 \leq n < m+N)$, $\mu_n = \beta \min (m, n)$, has been studied by Kühn (1971) and Rosenlund (1975), with separate methods and partly separate results. Writing the latter article we were not aware of the priority of Kühn, which is hereby acknowledged.

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