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THE RANDOM ORDER SERVICE G/M/m QUEUE

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ABSTRACT

The waiting time in the random order service $G/M/m$ queue is studied. For the Laplace transform we obtain a simpler representation than previously available. For the moments, an explicit recursive algorithm is given and carried out numerically for some cases. This gives rise to the conjecture that the waiting-time distribution can be approximated by the one for $M/M/m$ after a suitable change of scale.

1. INTRODUCTION

Consider the random order service $G/M/m$ queue with interarrival time distribution function $F(F(0)=0)$ and service time density $\mu e^{-\mu t}$. Let G be the distribution function of the waiting time W (in the stationary case) conditional on its being positive, i.e., $G(t) = P(W \leqslant t \mid W>0)$. We shall study G through the Laplace transform $\hat{G}(s) = \int_0^\infty e^{-st} dG(t)$ and the moments $\mu_n = \int_0^\infty t^n dG(t)$ of G.

The model has been investigated by LeGall [3] and Takács [6], who give the characteristic function and Laplace transform, respectively, of W. Carter and Cooper [1] and Cooper [2] study G directly and give recursive algorithms for its computation. Carter and Cooper [1] mention that their analysis was motivated by a study of the Bell System's No. 101 Electronic Switching System.

By a substitution of function in Takács' basic differential Eq. (27) we are here able to obtain a simpler closed expression for the Laplace transform than that following from Takács' Eq. (23). Also we give a simpler recursive algorithm for the moments than the one indicated by Takács in his Eq. (32)-(36). The algorithm is carried out numerically for some special cases. The study of moments gives rise to some conjectures on approximations for G.

The $G/M/m/N$ model has been studied by Rosenlund ([5] § 10). By relations (20), (24), (27), (28) and (29) the Laplace transform of W can for $N<\infty$ be calculated without need for numerical integration in the usual $D/M/m$ and $E_K/M/m$ cases, so that it might be preferable to approximate the present infinite waiting room with a finite waiting room. Our notation is

$$\beta = m\mu,$$

$$\hat{F}(s) = \int_0^\infty e^{-st} dF(t) ,$$

$$\psi(s) = \hat{F}(\beta s),$$

$\rho = (\beta \int_0^\infty t\, dF(t))^{-1}$ (assumed < 1),

$\alpha(s) =$ root z with smallest absolute value of the equation $z = \psi(s + 1 - z)$,

$\omega = \alpha(0)$,

$A =$ as given by (11) in Takács [6] (cited by Cooper ([2], p. 186)),

$W_i =$ waiting time of i th customer,

$V_s(x) = \sum_{k=0}^{\infty} E(e^{-s\beta W_i} | i$ th customer finds $m + k$ other customers at arrival x^k,

$M_n(x) = \sum_{k=0}^{\infty} E((\beta W_i)^n | m + k$ customers before i th arrival$) x^k$,

$W =$ random variable with the limiting distribution for W_i as $i \to \infty$.

The notation is adapted to obtain functions which are invariant under changes of time scale.

Let P_k be an arriving customer's distribution for the number of other customers in the system in the stationary (long-run) case. From Takács [6] Eq. (9), we quote

$$P_k = A\omega^{k-m}, k \geqslant m.$$

Hence we can derive the relations

(1) $$P(W \leqslant t) = 1 - A(1-\omega)^{-1} + A(1-\omega)^{-1} G(t),$$

(2) $$\hat{G}(s) = (1-\omega) V_{s/\beta}(\omega),$$

(3) $$E(e^{-sW}) = 1 - A(1-\omega)^{-1} + A(1-\omega)^{-1}\hat{G}(s),$$

(4) $$\mu_n = (1-\omega)\beta^{-n} M_n(\omega),$$

(5) $$E(W^n) = A(1-\omega)^{-1}\mu_n.$$

For $m = 1$, it holds that $A = \omega(1-\omega)$.

2. THE LAPLACE TRANSFORM

From Takács [6] Eq. (27), we get

(6) $$(x - \psi(s+1-x)) V'_s(x) + V_s(x) = (1 - \psi(s+1-x))(1-x)^{-1}(s+1-x)^{-1}, \quad 0 \leqslant x < 1.$$

The relation between Takács' notation and ours is $\hat{F}(s) = \phi(s)$, $\alpha(s) = \gamma(\beta s)$ and $V_s(x) = \Phi(\beta s, x)$. Equation (6) also follows from Eq. (24) in Rosenlund [5], which was derived by different methods and is in a different form than Eq. (26) in Takács [6]. Before its solution we make the substitution

$$U_s(x) = V_s(x) - (s + 1 - x)^{-1}.$$

Then from Eq. (6)

$$U'_s(x) + U_s(x)/(x - \psi(s+1-x)) = s(1-x)^{-1}(s+1-x)^{-2}.$$

Now take $s > 0$ real and let I stand for either $[0, \alpha(s))$ or $(\alpha(s), 1)$. With z an arbitrary fixed point in I we have for x in I

$$\frac{d}{dx}\left[U_s(x)\exp\left\{\int_x^z \frac{dt}{\psi(s+1-t)-t}\right\}\right] = s(1-x)^{-1}(s+1-x)^{-2} \exp\left\{\int_x^z \frac{dt}{\psi(s+1-t)-t}\right\},$$

whence

$$U_s(x)\exp\left\{\int_x^z \frac{dt}{\psi(s+1-t)-t}\right\} = \int_z^x s(1-u)^{-1}(s+1-u)^{-2}\exp\left\{\int_u^z \frac{dt}{\psi(s+1-t)-t}\right\}du + C_{s,z}. \tag{7}$$

Setting $x = z$ it is seen that the constant of integration $C_{s,z} = U_s(z)$. Let now $x \to \alpha(s)$. Then $\psi(s+1-x) - x \sim (\alpha(s) - x)(1 + \psi'(s+1-\alpha(s))$, so that the left side of Eq. (7) tends to 0. Hence the first term of the right side is $-U_s(z)$ for $x = \alpha(s)$. Put $Q(s) = U_s(\omega)/s$. Then

$$Q(s) = \int_{\alpha(s)}^{\omega} (1-u)^{-1}(s+1-u)^{-2}\exp\left\{\int_u^{\omega} \frac{dt}{\psi(s+1-t)-t}\right\}du. \tag{8}$$

The exp factor is $\leqslant 1$. A comparison with Eq. (23) in Takács [6] reveals the relative simplicity of Eq. (8). From Eq. (2) we now get an expression for $\hat{G}(s)$. Making substitutions of variable to get real intervals of integration also for complex s we obtain, letting

$$f_s(y) = 1 - \alpha(s) - (\omega - \alpha(s))y,$$

$$Q(s) = (\omega - \alpha(s))\int_0^1 f_s(y)^{-1}(s + f_s(y))^{-2} \exp\left\{\int_y^1 \frac{(\omega - \alpha(s))\,dt}{\psi(s + f_s(t)) + f_s(t) - 1}\right\}dy. \tag{9}$$

The resulting expression for $\hat{G}(s)$ holds also for complex s with $Re(s) \geqslant 0$, and we can use Lévy's inversion formula for characteristic functions, which for distribution functions F such that $F(t) = 0$ for $t < 0$ can be written

$$F(t) = \frac{2}{\pi}\int_0^{\infty} \sin(tx)x^{-1}\,\mathrm{Re}(\hat{F}(ix))\,dx, \tag{10}$$

if $t \geqslant 0$ is a point of continuity for F. The integral is defined at least in the improper Riemann sense. Inverting $\hat{G}(s)$ we note that $(\beta - \beta\omega)/(s + \beta - \beta\omega)$ is the Laplace transform of the exponential distribution with mean $1/(\beta - \beta\omega)$. This is the distribution of waiting time (conditional on its being positive) in the "first come, first served" $G/M/m$ queue. See Eq. (14) in Takács [6]. We can now state

THEOREM 1: With Q given by Eq. (8) or (9) it holds that

$$\hat{G}(s) = (\beta - \beta\omega)/(s + \beta - \beta\omega) + (1 - \omega)(s/\beta)Q(s/\beta)$$

and

$$G(t) = 1 - e^{-(1-\omega)\beta t} - \frac{2}{\pi}(1-\omega)\int_0^\infty \mathrm{Im}(Q(ix))\sin(\beta t x)dx.$$

From Eq. (7) we get

(11) $$M_1(x) = -\frac{\partial}{\partial s}\left\{V_s(x)\right\}_{s=0} = (1-x)^{-2} - \int_\omega^x (1-u)^{-3}$$
$$\exp\left\{\int_u^x \frac{dt}{\psi(1-t)-t}\right\}du.$$

This relation can be used for calculating mean wait when the arriving customer's queue length distribution is not the stationary one. Applying a Tauberian result we can obtain $G'(0) = \lim_{s\to\infty} s\hat{G}(s)$ from Theorem 1. By dominated convergence in Eq. (8) we obtain

(12) $$G'(0) = (\beta - \beta\omega)\omega^{-1}\log(1/(1-\omega)).$$

3. THE MOMENTS

Takács [6] indicates by his Eqs. (32)-(36) a method of calculating the moments $E(W^n)$. We shall here develop a simple and explicit recursive algorithm for this purpose. It is easily shown that

(13) $$M_n(x) = (-1)^n \frac{\partial^n}{\partial s^n}\left\{V_s(x)\right\}_{s=0}.$$

Let us differentiate both sides of Eq. (6) n times in s and r times in x, putting $s = 0$ and $x = \omega$. Simplifying the resulting equation by substituting

$$c_0 = 0,$$
$$c_r = (1-\omega)^{r-1}(-1)^r\psi^{(r)}(1-\omega)/r! \text{ for } r \geqslant 1,$$
$$B_{n,r} = (1-\omega)^{n+r+1}M_n^{(r)}(\omega)/(n!r!),$$

we obtain the following formula, which might be considered the most useful result of this note:

(14) $$B_{n,r} = (1 + r - rc_1)^{-1}\left\{\binom{n+r+1}{r} - rc_1B_{n,r} + \sum_{i=1}^{n}\sum_{k=0}^{r} \binom{n-i+r-k}{n-i}c_{n-i+r-k}\left[(k+1)B_{i,k+1} - \binom{i+k+1}{k}\right]\right\}, \quad n \geqslant 1,\ r \geqslant 0.$$

The terms with $B_{n,r}$ on the right side cancel out, and the term with $B_{n,r+1}$ vanishes, since $c_0 = 0$. Hence Eq. (14) is a recursion. In programming, no regard need be given to the term $-rc_1B_{n,r}$ provided all data registers for the B:s are zero initially. To get $B_{1,0}, B_{2,0}, \ldots, B_{p,0}$ we calculate Eq. (14) for $r = 0, \ldots, p-n$ and $n = 1, \ldots, p$. We get successively

$$B_{1,0}\ B_{1,1}\ \ldots\ B_{1,p-2}\ B_{1,p-1},$$
$$B_{2,0}\ B_{2,1}\ \ldots\ B_{2,p-2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$B_{p-1,0}\ B_{p-1,1},$$
$$B_{p,0}.$$

We need only $c_1, \ldots, c_{p-1}$. For $r > 0$ the interest of $B_{n,r}$ is only as a stepping stone on the way to $B_{1,0}, \ldots, B_{p,0}$. From Eq. (4) then

THEOREM 2: The moments of G are $\mu_n = (\beta - \beta\omega)^{-n} n! B_{n,0}$, where $B_{n,0}$ are obtained recursively from Eq. (14).

Note that the factor $(\beta - \beta\omega)^{-n} n!$ is the nth moment of the conditional waiting time distribution for first come, first served queue mentioned in connection with Theorem 1. Hence $B_{n,0}$ has independent interest as a comparison between disciplines of service with respect to moments.

The recursion (14) is well suited for numerical computation (see Table 1) but to throw more light on the mathematical form of μ_n we go further. Define

$$a_{1,0} = 1,$$
$$e_j = 1 + j - jc_1,$$
$$a_{n,r} = \prod_{j=1}^{n+r-1} e_j^{\min(n, n+r-j)},$$
$$D_{n,r} = a_{n,r} B_{n,r}.$$

Substitution in Eq. (14) gives

$$D_{n,r} = \binom{n+r+1}{r} e_r^{-1} a_{n,r} - r c_1 e_r^{-1} D_{n,r} + \sum_{i=1}^{n} \sum_{k=0}^{r} \tag{15}$$
$$\binom{n-i+r-k}{n-i} c_{n-i+r-k} \left[(k+1) e_r^{-1} a_{n,r} a_{i,k+1}^{-1} D_{i,k+1} - \binom{i+k+1}{k} e_r^{-1} a_{n,r} \right].$$

As before, the terms with $D_{n,r}$ and $D_{n,r+1}$ on the right side cancel out and vanish, respectively. For all other terms the coefficient $e_r^{-1} a_{n,r} a_{i,k+1}^{-1}$ can be seen to be a polynomial in $e_1, \ldots, e_{n+r-1}$ and hence in c_1. Thus $D_{n,r}$ is a polynomial in $c_1, \ldots, c_{n+r-1}$.

THEOREM 3: It holds that $\mu_n = (\beta - \beta\omega)^{-n} n! D_{n,0} / \prod_{j=1}^{n-1} (1 + j - jc_1)^{n-j}$, where $D_{n,0}$ is a polynomial in $c_1, \ldots, c_{n-1}$ obtained recursively from Eq. (15).

For the first three moments we obtain

$$\begin{cases} D_{1,0} = 1 \\ D_{2,0} = 2 \\ D_{3,0} = c_2(6 - c_1) + 12 - 8c_1 + 3c_1^2 - c_1^3. \end{cases} \tag{16}$$

Takács [6] gave the first and the second moment. For $n \geqslant 4$ the closed expressions for $D_{n,0}$ are complicated, and the recursion (14) is preferable for obtaining numerical results.

Let us apply the results to the cases of constant and Erlang-distributed interarrival times. For the deterministic case, where $\hat{F}(s) = e^{-sT}$ and $\rho = 1/\beta T$, we define ω by $\omega = \exp\{(\omega - 1)/\rho\}$, $0 < \omega < 1$. It holds that

$$c_r = \omega(1-\omega)^{-1}(-\log(\omega))^r/r!, \; r \geqslant 1. \tag{17}$$

For the gamma (Erlang) case, where $\hat{F}(s) = (\lambda/(s+\lambda))^K$ $(K > 0)$ and $\rho = \lambda/\beta K$, ω is defined by $\omega = (1+(1-\omega)/\rho K)^{-K}$, $0 < \omega < 1$. Here

$$c_r = \omega(1-\omega)^{-1}(1-\omega^{1/K})^r \binom{K-1+r}{r}, \; r \geqslant 1. \tag{18}$$

In particular for the M/M/m queue $(K = 1)$ we have $\omega = \rho = \lambda/\beta$ and $c_r = (1-\omega)^{r-1}\omega$ $(r \geqslant 1)$. It follows that for this case

$$\mu_3 = 12(\beta - \lambda)^{-3}(2 + \omega)(2 - \omega)^{-2}. \tag{19}$$

Table 1 gives $B_{1,0}, B_{2,0}, \ldots, B_{10,0}$ for $\hat{F}(s)$ equal to e^{-sT}, $(\lambda/(s+\lambda))^4$, and $\lambda/(s+\lambda)$, i.e., for the queues $D/M/m$, $E_4/M/m$, and $M/M/m$, and for the traffic intensities $\rho = 0.5, 0.7$, and 0.9. We used the calculator TI 59 and run time was 3.75 h for each case, in all 33.75 h.

TABLE 1 — *Values of* $B_{n,0}$ *for* $1 \leqslant n \leqslant 10$

	$D/M/m$			$E_4/M/m$			$M/M/m$		
ρ	0.5	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
ω	.203188	.466996	.806900	.301931	.552912	.843335	.5	.7	.9
1	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
2	1.25500	1.50053	1.81251	1.28835	1.51643	1.81482	1.33333	1.53846	1.81818
3	1.97820	3.06019	4.76979	2.07762	3.11494	4.77927	2.22222	3.19527	4.79339
4	3.68889	7.69510	16.2385	3.95696	7.89274	16.2857	4.37037	8.18958	16.3561
5	7.75822	22.5330	67.1263	8.51405	23.3153	67.3971	9.72840	24.5066	67.8014
6	17.8677	74.1797	323.777	20.1019	77.5100	325.506	23.8214	82.6458	328.092
7	44.2026	268.155	1773.23	51.0705	283.195	1785.31	62.9102	306.694	1803.40
8	115.889	1046.83	10810.9	137.706	1118.21	10902.1	176.680	1231.30	11038.9
9	318.837	4359.09	72282.5	390.125	4712.69	73019.7	522.226	5281.22	74127.5
10	913.672	19180.3	523851	1152.41	20999.0	530185	1611.84	23968.8	539726

The table illustrates the heavy tail of G in comparison with that of the exponential distribution with mean $1/(\beta - \beta\omega)$, particularly under heavy traffic.

Holding F fixed up to a scale factor and letting $\rho \to 1$ we have $\lim c_1 = 1$ and $\lim c_r = 0$ for $r \neq 1$. This gives in Eq. (14)

$$\lim_{\rho \to 1} B_{n,r} = (n+r)!/r!, \tag{20}$$

so that by Theorem 2

$$\lim_{\rho \to 1} (\beta - \beta\omega)^n \mu_n = (n!)^2. \tag{21}$$

Now $(n!)^2 = E((XY)^n)$, where X and Y are independent with density e^{-x}. We cannot, however, deduce that $\lim G(t/(\beta - \beta\omega)) = P(XY \leqslant t) = \int_0^\infty e^{-y}(1-e^{-t/y})dy$ since the moment sequence $(n!)^2$ does not determine the corresponding distribution uniquely. Neither does it seem possible to establish such a convergence from the Laplace transform $\hat{G}$.

Even for moderately heavy traffic Table 1 reveals that we can write approximately $B_{n,0} \approx (n!)^{\alpha}$ for some $\alpha (0 < \alpha < 1)$ determined by the parameters, so that

$$\mu_n \approx (\beta - \beta\omega)^{-n}(n!)^{1+\alpha}. \tag{22}$$

We can determine α to make Eq. (22) exact for $n = 3$, i.e.,

$$\alpha = \log(B_{3,0})/\log(6), \tag{23}$$

where, by Eq. (16) and Theorem 3,

$$B_{3,0} = [c_2(6 - c_1) + 12 - 8c_1 + 3c_1^2 - c_1^3](2 - c_1)^{-2}(3 - 2c_1)^{-1}.$$

The approximation is not so good for light traffic.

It is further seen from Table 1 that $B_{n,0}$ depends heavily on the traffic intensity ρ but not much, given ρ, on the form of the interarrival distribution F, although μ_n depends strongly on F through ω in the factor $(\beta - \beta\omega)^{-n}$. This suggests that G can be approximated by the distribution for $M/M/m$ after a change of scale. More precisely, let $G_{\lambda,\beta}$ denote G when $F(t) = 1 - e^{-\lambda t}$; then our results would indicate the approximation

$$G(t) \approx G_{\rho,1}(t(\beta - \beta\omega)/(1 - \rho)) \tag{24}$$

for ρ not too small. The indeterminancy of the moment problem still remains, though.

4. THE DISTRIBUTION FUNCTION G FOR M/M/m AND D/M/m

For the queues $M/M/m$ and $D/M/m$, special formulas give more useful results for the calculation of G than the general inversion formula of Theorem 1. For $M/M/m$ the result of Pollaczek [4] seems to be the most convenient. It can be written

$$G_{\rho,1}(t) = 1 - 2(1 - \rho) \int_0^{\pi} \frac{\exp\left\{\left[x + 2\arctan\left(\frac{\sqrt{\rho}\sin x}{1 - \sqrt{\rho}\cos x}\right)\right]\cot x - t(1 + \rho - 2\sqrt{\rho}\cos x)\right\}\sin x \, dx}{(1 + \rho - 2\sqrt{\rho}\cos x)^2(1 + e^{\pi \cot x})}. \tag{25}$$

For $D/M/m$ the most convenient algorithm seems to be the so-called additional conditioning variable method due to P. J. Burke, described in Cooper [2], pp 229-230. In this case the conditioning variable, the number of arriving customers in $(0,t)$, is deterministic. The algorithm is a recursive scheme, which for $D/M/m$ can be reformulated in the following way. Let

$$H_{j,k}(t) = P(\beta W_i > k/\rho + t \mid m + j \text{ customers before } i\text{th arrival}), \quad k = 0, 1, \ldots; 0 \leqslant t \leqslant \rho^{-1}. \tag{26}$$

Then

$$G(t) = 1 - (1 - \omega)\sum_{j=0}^{\infty} \omega^j H_{j,[\beta\rho t]}(\beta t - [\beta\rho t]/\rho), \; t \geqslant 0, \tag{27}$$

and $H_{j,k}$ is determined by the recursion

$$(28)\quad \begin{cases} H_{j,0}(t) = \sum_{r=1}^{j+1} \frac{r}{j+1} \frac{t^{j+1-r}}{(j+1-r)!} e^{-t}, j \geq 0 \\ H_{j,k}(t) = \sum_{r=1}^{j+1} \frac{r}{j+1} \frac{(1/\rho)^{j+1-r}}{(j+1-r)!} e^{-1/\rho} H_{r,k-1}(t),\ k \geq 1; j \geq 0. \end{cases}$$

Forming the power series

$$\overline{H}_{k,t}(x) = \sum_{j=0}^{\infty} x^j H_{j,k}(t),\ 0 \leq x < 1,$$

we have

$$(29)\quad G(t) = 1-(1-\omega)\overline{H}_{[\beta\rho t],\beta t - [\beta\rho t]/\rho}(\omega).$$

From Eqs. (28) we obtain the recursion

$$(30)\quad \begin{cases} \overline{H}_{0,t}(x) = x^{-1} \int_0^x e^{t(u-1)}(1-u)^{-2} du \\ \overline{H}_{k,t}(x) = x^{-1} \int_0^x e^{(u-1)/\rho} \overline{H}'_{k-1,t}(u) du,\ k \geq 1. \end{cases}$$

This results in

$$(31)\quad G(t) = 1 - (1-\omega)\omega^{-1} \int_{1-\omega}^{1} e^{-\beta t u} u^{-2} du, 0 \leq t \leq T.$$

The formula will hold for any arrival distribution F such that $F(T-0) = 0$.

For numerical computations it appears that Eq. (31), when applicable, is better than Eqs. (27) and (28), while for $t > T(\beta t > \rho^{-1})$ generally Eqs. (27) and (28) are better than Eqs. (29) and (30). In Table 2 we study the suggested approximation (24). For each ρ-value, the

TABLE 2 — *Values of G for D/M/m with approximation (24)*

	$\rho = 0.5$		$\rho = 0.8$	
βt	$G(t)$	appr.	$G(t)$	appr.
0	0	0	0	0
0.25	0.1995	0.2315	0.1353	0.1580
0.50	0.3591	0.3962	0.2510	0.2744
0.75	0.4868	0.5169	0.3501	0.3646
1.0	0.5889	0.6077	0.4352	0.4369
1.25	0.6707	0.6775	0.5084	0.4964
1.50	0.7371	0.7322	0.5483	0.5464
1.75	0.7885	0.7756	0.5872	0.5889
2.0	0.8305	0.8106	0.6244	0.6257
2.5	0.8683	0.8624	0.6921	0.6858
3.0	0.9013	0.8981	0.7331	0.7328
4.0	0.9482	0.9413	0.8028	0.8011
5.0	0.9670	0.9647	0.8503	0.8477
7.0	0.9872	0.9859	0.9060	0.9054
10.0	0.9965	0.9958	0.9499	0.9491
17.0	0.9997	0.9996	0.9848	0.9846

left column gives $G(t)$ for $D/M/m$ while the right column gives $G_{\rho,1}(\beta t(1-\omega)/(1-\rho))$. At least for the larger values of the argument the agreement is seen to be good for both traffic intensities. Since $D/M/m$ might be denoted $E_\infty/M/m$ and since the agreement in moments was shown to be better between $E_4/M/m$ and $M/M/m$ than between $D/M/m$ and $M/M/m$, the approximation (24) should be still better for $E_K/M/m$.

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