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WAITING FOR CLUSTERS IN INHIBITED RENEWAL PROCESSES

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1. Introduction

We will in this paper work with the interaction of two random point processes such that points in a renewal point process, the *excitatory process*, are eliminated by points in an independent *inhibitory process*. Such a thinning situation was suggested by M. TEN HOOPEN and H. A. REUVER in [3] as a mathematical model for neuron firing. In [1] it is discussed how this thinning procedure applies to certain stochastic service systems. Various aspects of this kind of interaction of renewal processes is treated at length in [2]. Our interest in this paper is in the *response process* of those retained points of the excitatory process which complete a *cluster* of retained points, which are near each other in some sense. We will first considers cluster of k successive retained points with no eliminated points between them. Such clusters were considered in [2] for some simple elimination schemes. We will here extend and modify the treatment in [2]. We will also prove a limit theorem for $k\to\infty$. We will furthermore generalize to clusters of k retained points, such that at most r points have been eliminated between any two successive points. Finally, we will mention some applications.

2. Waiting for a Cluster

We assume that the excitatory process is a discrete time renewal process. Let r^g be the geometric transform (or probability generating function) of its interarrival distribution, $r^g(s)=\sum_{k=1}^{\infty} r_k s^k$. Furthermore we assume that the inhibitory process is a ϱ-binomial process, independent of the excitatory process: that is, at each discrete time epoch a point of the inhibitory process is realized with probability $\varrho=1-\varphi$, independently of what happens at other time epochs.

The elimination takes place according to the following scheme. Assume that the points of the excitatory process are realized at the times $u_1, u_2, \dots$. If one or more inhibitory points arrive at times $1, \dots, u_1$, the next j excitatory points starting with the one at u_1 will be eliminated with probability p_j, where $j\in\{0, 1, \dots\}$. Note that p_0 is the probability that the inhibitors have no elimination effect. For $n\geqq 2$, assume that the point at u_n has not been eliminated by inhibitory points arriving at times $1, \dots, u_{n-1}$. If one or more inhibitory points arrive at times $u_{n-1}+1, \dots, u_n$, the next j excitatory points starting with the one at u_n will be eliminated with probability p_j. A point of the inhibitory process has no elimination effect if a simultaneous or subsequent excitatory point already has been eliminated by previous inhibitors.

We are in this section interested in the response process consisting of those points of the excitatory process, which satisfy the following condition. Each such point is the last point in a sequence of k successive points of the excitatory process, none of which has been eliminated and of which only the last one is a point of the response process. Here k is a fixed positive integer.

Let p_k^g be the geometric transform of the inter-arrival distribution of this embedded renewal process. In order to derive an expression for $p_k^g(s)$ we will use the collective marks (or the additional event) method. Thus we add a marking process to the situation such that each time epoch is marked independently with probability $1-s$, and not marked with probability s. This implies that $r^g(s)$ is the probability that no marks are given during the excitatory interval.

We now define probabilities $d_0(s)$ and $d_1(s)$ as follows:

$d_0(s)$=the probability that no marks are given during the first excitatory interval $[1, u_1]$ and that the point at the end of this interval is not eliminated,

$d_1(s)$=the probability that the inhibitors arriving during the first excitatory interval eliminate one or more consecutive points of the excitatory process, and that up to and including the last elimination no marks have been given.

We now get by proper conditioning

$$d_0(s) = \sum_{k=1}^{\infty} r_k s^k\big(\varphi^k+(1-\varphi^k)p_0\big) = r^g(\varphi s)+p_0\big(r^g(s)-r^g(\varphi s)\big). \tag{2.1}$$

Let furthermore $d_{1,j}(s)$ be the probability that no marks are given during the first excitatory interval and that the next j points are eliminated. (Time epochs after the first excitatory interval may or may not be marked.) Then $d_1(s)=\sum_{j=1}^{\infty} d_{1,j}(s)r^g(s)^{j-1}$. We have for $j\in\{1, 2, \dots\}$

$$d_{1,j}(s) = \sum_{k=1}^{\infty} r_k s^k(1-\varphi^k)p_j = p_j\big(r^g(s)-r^g(\varphi s)\big),$$

from which it follows with $p^g(s)=\sum_{j=0}^{\infty} p_j s^j$

$$d_1(s) = r^g(s)^{-1}\big(r^g(s)-r^g(\varphi s)\big)\big(p^g(r^g(s))-p_0\big). \tag{2.2}$$

If we interprete $p_k^g(s)$ as the probability of no marks during the waiting time for the first point of the response process, we get by conditioning

$$p_k^g(s) = d_0(s)^k+\sum_{i=0}^{k-1} d_0(s)^i d_1(s)p_k^g(s),$$

which gives

$$p_k^g(s) = \frac{d_0(s)^k(1-d_0(s))}{1-d_0(s)-d_1(s)\big(1-d_0(s)^k\big)}.$$

This result can be written as

$$(2.3)\qquad 1-p_k^g(s)=\big(1-d(s)\big)\cdot\frac{1-d_0(s)^k}{1-d(s)+d_1(s)d_0(s)^k},$$

where $d(s)=d_0(s)+d_1(s)$. By dividing with $1-s$ and letting $s\to 1$ we get an expression for the expectation $\mu_k=p_k^{g\prime}(1)$. With

$$\mu=r^{g\prime}(1),\quad a=1-r^g(\varphi),\quad b=r^g(\varphi)$$

and

$$\alpha=\sum_{k=1}^{\infty}kp_k$$

we get

$$(2.4)\qquad \mu_k=\mu\big(b+a(\alpha+p_0)\big)\cdot\frac{1-(b+p_0a)^k}{a(1-p_0)(b+p_0a)^k}.$$

3. A limit theorem

In the previous section we have derived the probability distribution of the time intervals of the renewal response process. Let η_k be a random variable with this distribution.

THEOREM 1. *If* $p_0=0$, *then*

$$\lim_{k\to\infty}\mathsf{P}(\eta_k/\mu_k\leqq t)=1-e^{-t},\quad t\geqq 0.$$

PROOF. We observe that $\mu_k=\mu(b+a\alpha)(b^{-k}-1)/a=O(b^{-k})$. We let s depend on k in such a way that $s=e^{-wb^k}$ for $w\geqq 0$. We then get $\lim_{k\to\infty}d_0(s)=r^g(\varphi)=b$, $\lim_{k\to\infty}d_1(s)=1-b=a$. We can furthermore replace $d_0(s)^k=r^g(\varphi s)^k$ with b^k. This is justified as follows. Taylor expansions give

$$r^g(\varphi s)=b+(s-1)O(s),\quad (s\to 1)$$

and

$$s-1=e^{-wb^k}-1=-wb^kO(k),\quad (k\to\infty).$$

Thus

$$(r^g(\varphi s)/b)^k=(1-wb^{k-1}O(k))^k=\left(1+O\left(\frac{1}{k}\right)\right)^k\to 1,\quad (k\to\infty).$$

We rewrite (2.3) as

$$p_k^g(s)=\frac{1-d_0(s)}{\dfrac{1-d(s)}{d_0(s)^k}+d_1(s)}.$$

Since

$$\lim_{k\to\infty}\frac{1-d(s)}{d_0(s)^k}=\lim_{k\to\infty}\frac{1-d(s)}{1-s}\cdot\frac{1-e^{-wb^k}}{b^k}=d'(1)w=\mu(b+\alpha a)w,$$

we find that

$$\lim_{k\to\infty} p_k^g(e^{-wb^k}) = \frac{1-b}{1-b+\mu(b+\alpha a)w} = \frac{1}{1+\mu\dfrac{b+\alpha a}{a}w}.$$

It follows that the Laplace—Stieltjes transform of η_k/μ_k for $k\to\infty$ tends to $\dfrac{1}{1+w}$ and thus the theorem is proved.

4. A generalization

We will in this section consider an elimination scheme such that $p^g(s)=s$, that is, one or more consecutive points of the inhibitory process eliminate only the next point of the excitatory process. We are interested in a response process, which consists of those points of the excitatory process, that satisfy the following condition. The point is the last point in a sequence of k successive retained points such that between any two consecutive points at most r points of the excitatory process have been eliminated and such that none is already a point of the response process. The case studied in previous sections corresponds to $r=0$.

Let $p_{k,r}^g$ be the geometric transform of the interarrival distribution of this response process. We will derive an expression for $p_{k,r}^g(s)$, interpreted as the probability for no marking during the response interval. For that purpose we define $h_0(s)$ to be the probability that counting from a retained point at most r points are eliminated before the next retained point and that no marks are given in the interval between the two successive retained points. Then

$$h_0(s) = \sum_{i=0}^{r} [r^g(s)-r^g(\varphi s)]^i r^g(\varphi s) = \frac{1-[r^g(s)-r^g(\varphi s)]^{r+1}}{1-r^g(s)+r^g(\varphi s)} r^g(\varphi s).$$

Let furthermore $h_1(s)$ be the probability that at least $(r+1)$ points are eliminated between two retained points and no marks given. Then

$$h_1(s) = [r^g(s)-r^g(\varphi s)]^{r+1}.$$

By proper conditioning we obtain

$$p_{k,r}^g(s) = r^g(\varphi s)h_0(s)^{k-1} + [r^g(s)-r^g(\varphi s)]p_{k,r}^g(s) + \sum_{i=0}^{k-2} r^g(\varphi s)h_1(s)h_0(s)^i,$$

which gives

$$p_{k,r}^g(s) = \frac{r^g(\varphi s)h_0(s)^{k-1}[1-h_0(s)]}{1-r^g(s)+r^g(\varphi s)[r^g(s)-r^g(\varphi s)]^{r+1}h_0(s)^{k-1}}. \tag{4.1}$$

From this follows

(4.2) $$1-p^g_{k,r}(s) =$$

$$= \frac{1-r^g(s)}{1-r^g(s)+r^g(\varphi s)h_1(s)h_0(s)^{k-1}}\left[1+r^g(\varphi s)h_0(s)^{k-1}\frac{[r^g(s)-r^g(\varphi s)]^{r+1}-1}{1-r^g(s)+r^g(\varphi s)}\right],$$

which easily gives the following formula for the corresponding expectation $\mu_{k,r}$.

(4.3) $$\frac{\mu_{k,r}}{\mu} = \frac{1-c^k}{ba^{r+1}c^{k-1}},$$

where

$$a = 1-r^g(\varphi), \quad b = r^g(\varphi) \quad \text{and} \quad c = 1-a^{r+1}.$$

It follows from (4.3) that

$$\lim_{r\to\infty} p^g_{k,r}(s) = \left[\frac{r^g(\varphi s)}{1-r^g(s)+r^g(\varphi s)}\right]^k = p^g_r(s)^k,$$

where p^g_r is the geometric transform of the length of the time interval between successive retained points in this case. Let $\eta_{k,r}$ be a random variable with geometric transform $p^g_{k,r}$. We can then, with the aid of (4.1), prove the following theorem. The proof is similar to the proof of theorem 1.

THEOREM 2.

$$\lim_{k\to\infty} p(\eta_{k,r}/\mu_{k,r} \leqq t) = 1-e^{-t}, \quad t \geqq 0.$$

5. Some concluding remarks

We have here considered the interaction between two discrete time point processes. It is easy, by a proper limit process to derive from our results formulas for the case when an inhibitory Poisson process with intensity λ interacts with a general renewal point process with inter-arrival distribution F.

Our results can be applied to the neuron firing process as follows. Stimuli arrive to a neuron and some of them are eliminated by an inhibitory process. The neuron will fire when it receives k stimuli, which are near each other, for instance between any two successive retained stimuli at most r stimuli have been deleted.

It is shown in [2] how results concerning the interaction of a renewal excitatory process and a binomial inhibitory process can be applied to a B/G/1 conveyor system, where customers arrive according to a binomial process and the service times have a general discrete probability distribution determined by the geometric transform r^g. Furthermore there is a finite waiting room and when it is empty the server will take a unit from a storage. Now assume that there is place for k units in this storage and that it is refilled in the following way. Each time the server has taken more than r units in a row from the conveyer a mechanism is relased so that the storage is filled up to its full capacity. This implies that the only way the storage can be emptied is that the server takes k units from the storage and that between any two of these units he has taken at most r units from the conveyor. But then $p_{k,r}^{q}$ in section 4 above is the geometric transform of the length of time from an epoch when the storage is full and the service of a unit is to begin until it is empty for the first time.

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